SEQUENTIAL TEST FOR THE PARAMETER OF GENERALIZED MAXWELL DISTRIBUTION

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ABSTRACT

Sequential probability ratio test is developed for testing the hypothesis regarding the parameter of a Generalized Maxwell distribution. The expressions for the operating characteristics (OC) and average sample number (ASN) functions are derived. For the purpose of plotting the OC and ASN functions different approaches are used.

Keywords: Generalized Maxwell distribution, SPRT, OC and ASN functions, Newton-Raphson method.

1. INTRODUCTION

The theory of sequential analysis is began in about 1943 with the work of A.Wald and G.A.Barnard. Sequential analysis has been heavily dominated by the sequential probability ratio test (SPRT). SPRT for testing a simple hypothesis against a simple alternative is first time given by Wald (1947). He also obtained the expressions for the operating characteristics (OC) and average sample number (ASN) functions of the proposed sequential test. The SPRT has been applied by various authors in order to test the simple and composite hypothesis, related to various life testing probabilistic models. For reference, Epstein and Sobel (1955) applied the SPRT for testing the simple hypothesis regarding the mean of a one-parameter negative exponential distribution, Johnson (1966) dealt with the problem the of testing the hypothesis regarding the scale parameter of the Weibull distribution when the shape parameter is known and, Phatarfod (1971) devolved the SPRT for testing composite hypothesis for shape parameter of the gamma distribution, when scale parameter is known. Chaturvedi, Kumar and Kumar (2000) developed sequential test of simple and composite hypothesis regarding the parameters of a class of distributions representing various life testing models.

2. THE SET-UP OF THE PROBLEM:

Let the random variable (r.v.) X follows the generalized Maxwell distribution presented by the probabilistic density function (p.d.f.)

$$f(x;\mu,\theta)=4\pi^{-1/2}\theta^{-3/2}(x-\mu)^2\exp\{-(x-\mu)^2/\theta\}; \qquad x>\mu>0,\ \theta>0 \quad \dots (2.1)$$

Given a sequence of observations X_1, X_2, X_3,.........., from (2.1), suppose one wish to test the simple null hypothesis $H_0 : \theta = \theta_0$ against the simple alternative $H_1 : \theta = \theta_1 (\theta_1 > \theta_0)$. The expression for the OC and ASN function are obtained and their behavior is studied by graph plotting.

3. SPRT FOR TESTING THE HYPOTHSIS REGARDING 'μ'.

The SPRT for testing H_0 is defining as follows:
Let

$$Z_i = \ln\left\{\frac{f(X_i;\theta_1,\mu)}{f(X_i;\theta_0,\mu)}\right\}$$

$$= \ln\left(\frac{\theta_0}{\theta_1}\right) + \left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)(X_i - \mu)^2 . \qquad ...(3.1)$$

We choose two numbers A and B such that $0 < B < 1 < A$. At the n^{th} stage, accept H_0 if $\sum_{i=1}^{n} Z_i \le \ln B$, reject H_0 if $\sum_{i=1}^{n} Z_i \ge \ln A$, otherwise continue sampling by taking the $(n+1)^{th}$ observation. If $\alpha \in (0,1)$ *and* $\beta \in (0,1)$ are Type I and Type II errors, respectively, then according to Wald (1947), A and B are approximately given by

$$A \cong (1-\beta)/\alpha , \qquad B \cong \beta/(1-\alpha). \qquad \ldots(3.2)$$

The OC function of the SPRT is given by

$$L(\theta) \cong \frac{A^h - 1}{A^h - B^h} , \qquad \ldots(3.3)$$

Where 'h' is the non-zero solution of the equation

$$E\,[\,e^{hz}\,] = 1 \qquad \ldots(3.4)$$

From (1.1) and (3.1), since

$$E[e^{hz}] = \left\{1 - h\theta\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\right\}^{-1} \left(\frac{\theta_0}{\theta_1}\right)^h = 1 \qquad \ldots(3.5)$$

We obtained from (3.4) that

$$\theta = \frac{1-(\theta_0/\theta_1)^h}{h\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)}. \qquad \ldots(3.6)$$

The ASN function is approximately given by

$$E(N/\theta) = \frac{[L(\theta)\ln B + \{1-L(\theta)\}\ln A]}{E(Z)}, \qquad \ldots(3.7)$$

Provided E(Z) $\neq$ 0, where

$$\text{E(Z)} = \frac{3}{2}\{\ln((\theta_0/\theta_1) + \theta\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\}. \qquad \ldots(3.8)$$

From (3.7) the ASN function under is given, respectively by

$$E_0(N) \approx \frac{[(1-\alpha)\ln B + \alpha\ln A]}{\frac{3}{2}\{\ln(\theta_0/\theta_1) + \theta\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\}},$$

and

$$E_1(N) \approx \frac{[\beta\ln B + (1-\beta)\ln A]}{\frac{3}{2}\{\ln(\theta_0/\theta_1) + \theta\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\}}.$$

Now in order to plot OC and ASN functions, some method are provided below.

4. (A) METHOD FIRST (SIMULATION):

Consider the equation (3.3) and (3.6). For any arbitrarily chosen value ‘h’ the point [$\theta, L(\theta)$], computed from (3.3) and will be a point on the OC function.

The OC function can be drawn by plotting a sufficiently large number of points [$\theta, L(\theta)$] corresponding to various values of ‘h’ (see Wald 1947, p. 51).

REMARKS 1: We considered the testing of the hypothesis. $H_0 : \theta_0 = 25 \; versus \; H_1 : \theta_1 = 30$. fixing $\alpha = \beta = .05$. For equation (3.3) and (3.6) various pairs of [$\theta, L(\theta)$] are derived through varying ‘h’ between the interval (-2, 2). The interval is chose in such a way, so that the parameter θ lies between H_0 *and* H_1. The results are presented in Table 1. It is evident from the Table-1 that the values of $L(\theta_0) \, and \, L(\theta_1)$ are quite close to their theoretical values .95 and .05, respectively. The OC function curve and ASN function curve are plotted in Fig.1 and Fig.2.

(B) METHOD SECOND (APPROXIMATION):

Consider the equation (3.6) and taking the logarithms of both sides, using the expression

$$\ln(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} ...., \qquad -1 < x < 1$$

and retaining the up to third degree in ‘h’ we get

$$\frac{1}{3}\left\{\theta(\frac{1}{\theta_0} - \frac{1}{\theta_1})\right\}^3 h^2 + \frac{1}{2}\left\{\theta(\frac{1}{\theta_0} - \frac{1}{\theta_1})\right\}^2 h + \ln\left(\frac{\theta_0}{\theta_1}\right) + \theta(\frac{1}{\theta_0} - \frac{1}{\theta_1}) = 0$$

or

$$A_1 h^2 + B_1 h + C_1 = 0 \qquad ...(4.1)$$

Where

$$A_1 = \frac{1}{3}\left\{\theta(\frac{1}{\theta_0} - \frac{1}{\theta_1})\right\}^3$$

$$B_1 = \frac{1}{2}\left\{\theta(\frac{1}{\theta_0} - \frac{1}{\theta_1})\right\}^2$$

$$C_1 = \ln(\frac{\theta_0}{\theta_1}) + \theta(\frac{1}{\theta_0} - \frac{1}{\theta_1})$$

The solution of equation (4.1) is

$$h = \frac{-B_1 \pm \sqrt{B_1^{\ 2} - 4A_1C_1}}{2A_1} \qquad ...(4.2)$$

REMARKS 2: For testing the hypothesis $H_0 : \theta_0 = 25 \ \mathit{versus} \ H_1 : \theta_1 = 30$, fixing $\alpha = \beta = .05$ for different values of θ, the real roots h obtained from (4.2). The OC and ASN functions are evaluated with the help of (3.3) and (3.6) respectively. It is interesting to note that the values of 'h' obtained through approximations are given quite satisfactorily results (see Table 2). The OC and ASN functions are plotted in Fig.2.

(C) METHOD THIRD (NEWTON-RAPHSON):

$$F = \ln\left[1 - h\theta(\frac{1}{\theta_0} - \frac{1}{\theta_1})\right] - h\ln(\theta_0/\theta_1)$$

$$FD = \frac{1}{\left\{1 - h\theta(\frac{1}{\theta_0} - \frac{1}{\theta_1})\right\}}\left[\theta(\frac{1}{\theta_1} - \frac{1}{\theta_0})\right] - \ln(\theta_0/\theta_1)$$

FD is the first derivative of F.

The ASN function is approximately given by

$$E(N/\theta) = \frac{[L(\theta)\ln B + \{1 - L(\theta)\}\ln A]}{E(Z)},$$

where, $EZ = \frac{3}{2}\{\ln(\theta_0/\theta_1) + \theta(\frac{1}{\theta_0} - \frac{1}{\theta_1})\}$.

It is interesting to note the values of 'h' obtained through approximation and N-methods (by taking the approximate value of 'h' as initial value for N-R method) are quite close (the use of see Table). This justifies the use of approximation.

Table 1: Values of OC and ASN by **Method First(Simulation Method)**

$$(H_0 : \theta_0 = 25, \quad H_1 : \theta_1 = 30, \alpha = \beta = .05 \ ), \ and -2 \le h \le 2)$$

θ	$L(\theta)$	$E(N)$
22.91667	0.997238	66.075306
23.31381	0.995033	72.257979
23.72046	0.991085	79.716675
24.13686	0.984050	88.762946
24.56328	0.971621	99.725939
25.00000	0.950000	112.850578
25.44730	0.913374	128.058654
25.90547	0.854042	144.507928
26.37481	0.764548	160.041516
26.85562	0.643110	171.079397
27.85297	0.356890	166.970656
28.37016	0.235452	152.446420
28.90016	0.145958	134.343840
29.44331	0.086626	116.191829
30.00000	0.050000	99.933196
30.57059	0.028379	86.189011
31.15548	0.015950	74.870611
31.75507	0.008915	65.623981
32.36977	0.004967	58.053773

Table 2: Values of OC and ASN by Approximation and Newton-Raphson Method

$$(H_0 : \theta_0 = 25, \quad H_1 : \theta_1 = 30, \alpha = \beta = .05)$$

θ	By Approximation		By Newton- Raphson	
	$L(\theta)$	$E(N)$	$L(\theta)$	$E(N)$
23.0	0.997546327	67.383446	0.996872	67.292090
23.5	0.994498323	75.672130	0.993495	75.518568
24.0	0.988170893	85.859567	0.986800	85.618398
24.5	0.975597885	98.332454	0.973912	97.983835
25.0	0.951801894	113.302454	0.950000	112.850577
25.5	0.909376406	130.436315	0.907802	129.934696
26.0	0.83947596	148.277911	0.838472	147.839457
26.5	0.735933429	163.797249	0.735570	163.544821
27.0	0.602271508	172.948436	0.602240	172.895116
27.5	0.455469902	172.786712	0.455473	172.776030
28.0	0.318934896	163.596983	0.319130	163.420567
28.5	0.209410467	148.575803	0.210256	148.143516
29.0	0.131070762	131.530654	0.132848	130.896870
29.5	0.079308275	115.133415	0.081947	114.411299
30.0	0.046796438	100.644626	0.050000	99.933207
30.5	0.027008041	88.375580	0.030435	87.735274
31.0	0.015209739	78.177800	0.018577	77.634816
31.5	0.008289678	69.744339	0.011403	69.302681
32.0	0.004301037	62.752735	0.007051	62.404605

Figure 1: The OC Curve

$$(H_0 : \theta_0 = 25, \quad H_1 : \theta_1 = 30, \alpha = \beta = .05)$$

(Simulation Method)

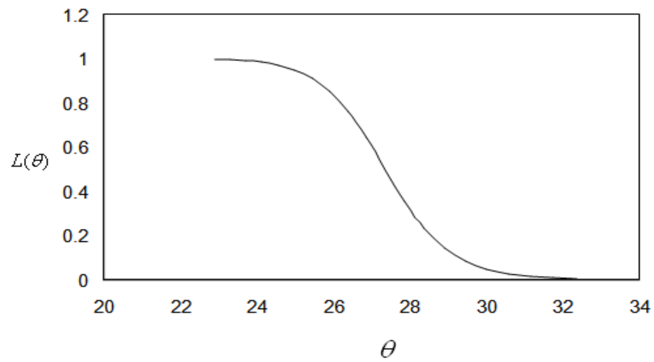


Figure 2: The ASN Curve

$$(H_0 : \theta_0 = 25, \quad H_1 : \theta_1 = 30, \alpha = \beta = .05)$$

(Simulation Method)

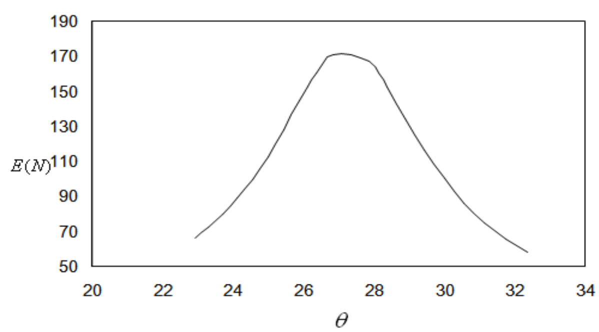


Figure 3: The OC Curve

$$(\ H_0 : \theta_0 = 25, \quad H_1 : \theta_1 = 30, \ \alpha = \beta = .05 \ )$$

Approximation Method and N-R Method

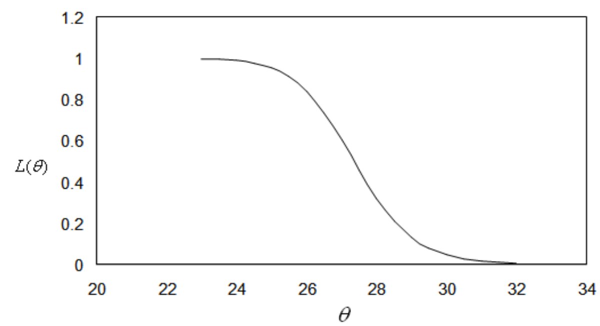
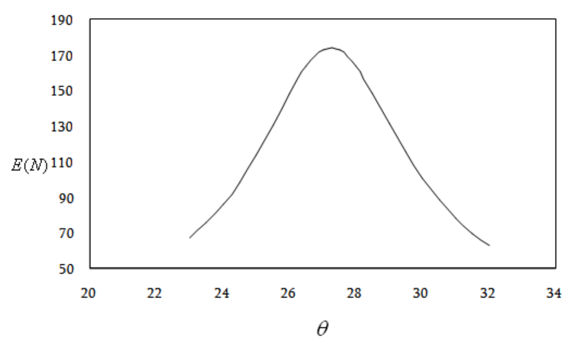


Figure 4: The ASN Curve

$$(\ H_0 : \theta_0 = 25, \quad H_1 : \theta_1 = 30, \ \alpha = \beta = .05 \ )$$

Approximation Method and N-R Method



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