Discrete-Time Hedging for European Contingent Claims via Risk Minimization in Market Measure

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Abstract

In this work, we propose a discrete-time hedging strategy for a European contingent claim (ECC) that reduces risk in the market measure. To this end, we minimize the second-moment of the hedging error in the market measure and give computable expressions for the positions that the seller must hold in the underlying asset (of the ECC) at any given hedge time. The minimization of the second-moment of the hedging error also yields an expression for the price of the option. The expressions obtained can be evaluated using Monte-Carlo methods. A noteworthy feature of the framework is that, it does not assume any specific model for the underlying asset price or involve any assumptions on the underlying market probability measure.

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Introduction

A *financial derivative* is a contract between two parties where the writer (seller) of the contract is obligated to deliver to the buyer, a payoff that is contingent upon the performance of the underlying assets. An example of an underlying is a stock, and that of a derivative is an option. If the payoff has to be delivered at a fixed time-to-maturity T, then such a derivative is called a European contingent claim (ECC). On the other hand, in the case of an American contingent claim (ACC), the buyer can exercise the contract anytime up to and including the time-to-maturity. In principle, the payoff of the contract could depend only on the asset value at the exercise time as in the case of simple options, or on the entire history of the asset value from the time of initiation of the contract to its exercise time as in the case of path-dependent options. The most popular example of a simple ECC is a European call option, where the seller is obligated to sell an asset to the buyer, at a particular strike value K, at expiration time T. Examples of path-dependent ECC's include Asian, barrier and lookback options. For an excellent introduction to the above topics, refer to (Hull, 2008), (Shreve, 2004), (Willmott, 2007).

Pricing and hedging of financial derivatives are fundamental problems in the field of mathematical finance and thus have been subjects of intense research for several decades. The problem of *pricing* a contract is to evaluate the value of the derivative at the time of its initiation. The price is the *premium* amount that the buyer would pay the seller. In general, the premium amount can vary depending upon the seller's perception of risk and on the pricing methodology used. In the case of a European call option, the celebrated work of Black and Scholes (Black & Scholes, 1973), and Merton (Merton, 1973), asserted that such contracts can be uniquely priced under certain market conditions. These conditions include continuous trading, absence of transaction costs, no restrictions on short selling and the assumption that the underlying asset price follows a Geometric Brownian motion (GBM).

The payoff of a contract is, typically, non-deterministic. As the seller of a contract has the obligation to deliver the payoff at the exercise time, he/she faces risk due to the random nature of the payoff. This risk has to be minimized and the process of minimizing the risk is *hedging*. The finer details of hedging depend on the nature of the contract, market conditions and the seller's perception of risk. For simple ECC's, when continuous trading is allowed, Black and Scholes (Black & Scholes, 1973) asserted the existence of a hedging strategy that can completely eliminate the risk of the seller by maintaining a *hedging* portfolio consisting of the underlying asset and riskless assets such as bond to replicate the market price of the derivative. For a lucid mathematical analysis on continuous trading, the reader can refer to (Harrison & Pliska, 1981).

While the work of Black and Scholes is path breaking, the assumption that one can readjust the portfolio in continuous time is unrealistic. In reality, a portfolio can only be adjusted a finite number of times over any finite period. A discrete-time strategy, at the best, can approximate its continuous time counterpart, only if, the number of hedging times are large. Therefore, with discrete time hedging, risk cannot be completely eliminated; it can only be minimized. In fact, Boyle and Emmanuel in (Boyle & Emmanuel, 1980) analysed the statistics of hedging errors and concluded that the magnitude of hedging errors increases with the time-discretization error. Hence, it is imperative to develop methodologies that minimize risk in some sense (see (Follmer & Sondermann, 1986), (Schweizer M. , 1992), (Schal, 1994), (Schweizer M. , 1995), (Mercurio & Vorst, 1996), (Angelini & Herzel, 2009) and references therein). Evidently, such strategies would depend upon the definition of risk.

In this paper, we measure risk in terms of the second-moment of the final money market position after the delivery of the payoff and propose an optimal hedging strategy that minimizes this risk, given a set of hedging times. In other words, we minimize the second-moment of the hedging error in the market measure. The strategy provides computable expressions for positions that the seller must hold in the underlying asset at any given hedge time. The framework also yields an expression for the price of the derivative that minimizes the second-moment. For the case of GBM, the positions have simple closed-form formulae. For a more general case, Monte-Carlo methods are to be employed to evaluate the expressions.

The outline of the paper is as follows. In the next section we provide a mathematical setting to the problem by introducing definitions and explaining our notations. Thereafter, we provide the expressions for the positions that a seller must hold, while adopting a hedging strategy that minimizes the second moment of the hedging error. Next, we elaborate on the relationship between the current strategy and a strategy that minimizes variance in the risk-neutral measure (Bhat, S.P. , Chellaboina, Kumar, Bhatia, & Prasad, 2009), (Bhat, S.P., Chellaboina, & Bhatia, 2010). Then we specialize our results to a European call option, after which numerical simulations are presented to illustrate the working of our strategy. Finally, we discuss the salient features of the proposed strategy and present our conclusions.

Mathematical Preliminaries

In this section, we introduce definitions, assumptions and notations to provide a mathematical setting for the problem. We consider a European contingent claim (ECC) written at time 0, with time-to-maturity T. We assume that $0 = T_0 < T_1 < T_2 <\ldots< T_n = T$ are the times at which portfolio readjustments are allowed. Denote S_k as the price of the underlying asset at time T_k, $k \in \{0,1,2,\ldots,n\}$, with a given S_0. Let $\{\Im_k\}_{k=0}^{n}$ denote a filtration on the probability space $(\Omega, \Im, \mathrm{P})$, where, for each $k \in \{0,1,2,\ldots,n\}$, $\Im_k$ is the σ -algebra on Ω generated by asset prices observed up to time T_k. Let $\Im = \Im_n$.

We make the following assumptions on the data. We assume that the return $R_k = (\bar{S}_k / \bar{S}_{k-1}) - 1$, over the interval $(T_{k-1}, T_k]$, $k \in \{1,2,\ldots,n\}$, is square integrable and is independent of $\Im_{k-1}$. Here $\bar{S}_k = e^{-rT_k} S_k$ is the discounted asset price process. The payoff V of the ECC is a square integrable random variable on $(\Omega, \Im, \mathrm{P})$.

Let Δ_k denote the amount of the underlying asset held in the hedging portfolio during $(T_{k-1}, T_k]$. At time T_n, the position Δ_n in the underlying asset is liquidated to provide the buyer the payoff V. The final money-market position of the portfolio, discounted to time to T_0, using the risk free interest rate r, is given by,

$$\begin{aligned}\overline{P} = \beta - \Delta_1 \overline{S}_0 - (\Delta_2 - \Delta_1)\overline{S}_1 - \cdots \\ - (\Delta_n - \Delta_{n-1})\overline{S}_{n-1} + \Delta_n \overline{S}_n - \overline{V}\end{aligned} \tag{1}$$

where β is the premium received for the ECC and $\overline{V} = e^{-rT}V$ is the discounted payoff of the ECC. The second-moment hedging problem is to seek $\Delta_k, k \in \{1,2,\ldots,n\}$, such that

1. Each Δ_k is a $\Im_{k-1}$ measurable random variable on Ω.
2. The second-moment of the discounted final money-market position $\overline{P}$, after delivering the payoff, under measure P, is minimum.

We conclude this section with some general comments on notation. We use $\Re$ to denote the set of real numbers, $\Re^{n\times m}$ to represent the set of $n \times m$ matrices and A^T to denote the transpose of the matrix A. For random variables X and Y, we write $X = Y$ to convey almost sure equality. The expectation operator is denoted by E. Occasionally, we write $\mathrm{E}^{\mathrm{P}}(\cdot)$ to stress that the expectation is taken under the measure P.

Second-Moment Minimizing Hedging Positions

Theorem 1: *The solution to the second-moment hedging problem is given by,*

$$\Delta_i^{\beta} = -\frac{1}{\overline{S}_{i-1} A_i^2}\begin{bmatrix} B_i \hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1} + B_i \beta \\ - \mathrm{E}(R_i X_i | \Im_{i-1}) \end{bmatrix}, \quad i \in \{1,2,\cdots,n\} \tag{2}$$

where

$A_i^2 = \mathrm{E}(R_i^2)$, $B_i = \mathrm{E}(R_i)$, $\hat{\Delta}_i = [\Delta_i, \Delta_{i-1}, \cdots, \Delta_1]^T$,

$\hat{\delta} S_i = [\overline{S}_i - \overline{S}_{i-1}, \overline{S}_{i-1} - \overline{S}_{i-2}, \cdots, \overline{S}_1 - \overline{S}_0]^T$

with,

$$X_n = \overline{V}$$
$$X_{i-1} = \frac{\mathrm{E}\left(\left(A_i^2 - B_i R_i\right) X_i \middle| \Im_{i-1}\right)}{A_i^2 - B_i^2}, i \in \{1,2,\cdots,n\} \qquad (3)$$

The premium β *that minimizes the second-moment of the hedging error is given by* X_0.

Proof: Let $i \in \{0,1,2,\cdots,n\}$. Define,

$$\overline{P}_i = \hat{\Delta}_i^T \hat{\delta} S_i + \left(\beta - X_i\right), \qquad (4)$$

where $\hat{\Delta}_i$ and $\hat{\delta} S_i$ are as defined in the statement of the theorem. Observe that, from equation (1) the discounted final money market position $\overline{P} = \overline{P}_n$. Denote $R_i = \left(\overline{S}_i - \overline{S}_{i-1}\right)/\overline{S}_{i-1}$ and rewrite equation (4) as

$$\overline{P}_i = \Delta_i \overline{S}_{i-1} R_i + \hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1} + \left(\beta - X_i\right), \qquad (5)$$

Squaring equation (5), we have,

$$\overline{P}_i^2 = \Delta_i^2 \overline{S}_{i-1}^2 R_i^2 + 2\Delta_i \overline{S}_{i-1} R_i \left(\hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1} + \left(\beta - X_i\right)\right) + \left(\hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1} + \left(\beta - X_i\right)\right)^2$$
$$= \Delta_i^2 \overline{S}_{i-1}^2 R_i^2 + 2\Delta_i \overline{S}_{i-1}\left(R_i \hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1} + R_i \beta - R_i X_i\right) + \left(\hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1} + \left(\beta - X_i\right)\right)^2$$

Taking expectation of $\overline{P}_i^2$ in the market measure and using the fact that for any random variable X, $\mathrm{E}\left(X\right) = \mathrm{E}\left(\mathrm{E}\left(X \middle| \Im_{i-1}\right)\right)$, we have,

$$E\left(\bar{P}_i^2\right) = E\left[\begin{array}{l} \Delta_i^2 \bar{S}_{i-1}^2 A_i^2 + 2\Delta_i \bar{S}_{i-1} \\ \left(B_i \hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1} + B_i \beta_i E\left(X_i \mid \Im_{i-1}\right) - R_i\right) \\ + \left(\hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1}\right)^2 - 2 S_{i-1}^T \hat{\delta}\, \hat{\Delta}_{i-1} \beta_i E\left(X_i \mid \Im_{i-1}\right) \\ + \beta_i^2 E\left(X_i^2 \mid \Im_{i-1}\right) - 2\beta_i E\left(X_i \mid \Im_{i-1}\right) \end{array}\right]$$

where we have used the independence of R_i to replace $E\left(R_i^2 \mid \Im_{i-1}\right)$ by $E\left(R_i^2\right) = A_i^2$ and $E\left(R_i \mid \Im_{i-1}\right)$ by $E\left(R_i\right) = B_i$. By completing the squares, we rewrite the above equation as,

$$E\left(\bar{P}_i^2\right) = E\left[\begin{array}{l} A_i^2\left(\Delta_i \bar{S}_{i-1} + \dfrac{B_i \hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1} + B_i \beta_i E\left(X_i \mid \Im_{i-1}\right) - R_i}{A_i^2}\right)^2 \\ - \dfrac{\left(B_i \hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1} + B_i \beta_i E\left(X_i \mid \Im_{i-1}\right)\right)^2}{A_i^2} \\ + \left(\hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1}\right)^2 - 2 S_{i-1}^T \hat{\delta}\, \hat{\Delta}_{i-1} \beta_i E\left(X_i \mid \Im_{i-1}\right) \\ + \beta^2 E\left(X_i^2 \mid \Im_{i-1}\right) - 2\beta E\left(X_i \mid \Im_{i-1}\right) \end{array}\right] \quad (6)$$

The arguments of the second expectation in equation (6) can be expanded as

$$\left(\hat{\Delta}_{i-1}^T \hat{\delta} S_{i-1}\right)^2 - 2 S_{i-1}^T \hat{\delta}\left(\frac{B_i^2}{A_i^2} - 1\right)\hat{\Delta}_{i-1}\beta_i E\left(X_i \mid \Im_{i-1}\right)$$

$$-\frac{1}{A_i^2}\left(B_i^2 \beta_i^2 E\left(X_i \mid \Im_{i-1}\right)^2\right)$$

$$+\left(\beta^2 E\left(X_i^2 \mid \Im_{i-1}\right) - 2\beta E\left(X_i \mid \Im_{i-1}\right)\right) \quad (7)$$

Using the fact that A_i^2 and B_i are constants, we rewrite the above equation (7)

$$= \left(1-\frac{B_i^2}{A_i^2}\right)\left(\hat{\Delta}_{i-1}^T\hat{\delta}_{i-1} - S_{i-1}\right)^2$$

$$+\,2\left(1-\frac{B_i^2}{A_i^2}\right)\left(\hat{\Delta}_{i-1}^T\hat{\delta}_{i-1} - S_{i-1}\right)\left(\beta - \frac{\mathrm{E}\left(\left(A_i^2 - B_iR_i\right)X_i \mid \Im_{i-1}\right)}{A_i^2 - B_i^2}\right)$$

$$+\left(1-\frac{B_i^2}{A_i^2}\right)\left(\beta^2 - 2\beta\frac{\mathrm{E}\left(\left(A_i^2 - B_iR_i\right)X_i \mid \Im_{i-1}\right)}{A_i^2 - B_i^2} + \left(\frac{\mathrm{E}\left(\left(A_i^2 - B_iR_i\right)X_i \mid \Im_{i-1}\right)}{A_i^2 - B_i^2}\right)^2\right)$$

$$-\frac{1}{A_i^2}\mathrm{E}\left(R_iX_i \mid \Im_{i-1}\right)^2 + \mathrm{E}\left(X_i^2 \mid \Im_{i-1}\right)$$

as,

$$-\left(1-\frac{B_i^2}{A_i^2}\right)\left(\frac{\mathrm{E}\left(\left(A_i^2 - B_iR_i\right)X_i \mid \Im_{i-1}\right)}{A_i^2 - B_i^2}\right)^2 \qquad (8)$$

Recalling that $X_{i-1} = \left(\frac{\mathrm{E}\left(\left(A_i^2 - B_iR_i\right)X_i \mid \Im_{i-1}\right)}{A_i^2 - B_i^2}\right)$ and expanding the last term of (8) we have,

$$= \left(1-\frac{B_i^2}{A_i^2}\right)\left(\hat{\Delta}_{i-1}^T\hat{\delta}_{i-1} + S_{i-1} - \mathrm{E}\beta\right)^2 + \mathrm{E}\left(X_i^2 \mid \Im_{i-1}\right) - \frac{1}{A_i^2}\mathrm{E}\left(R_iX_i \mid \Im_{i-1}\right)^2$$

$$-\frac{A_i^4\,\mathrm{E}\left(X_i \mid \Im_{i-1}\right)^2 - 2A_i^2B_i\,\mathrm{E}\left(X_i \mid \Im_{i-1}\right)\mathrm{E}\left(R_iX_i \mid \Im_{i-1}\right) + B_i^2\,\mathrm{E}\left(R_iX_i \mid \Im_{i-1}\right)^2}{\left(A_i^2 - B_i^2\right)A_i^2}$$

$$= \left(1-\frac{B_i^2}{A_i^2}\right)\bar{P}_{i-1}^2 + \mathrm{E}\left(X_i^2 \mid \Im_{i-1}\right) - \frac{A_i^2}{A_i^2 - B_i^2}\mathrm{E}\left(X_i \mid \Im_{i-1}\right)^2$$

$$+\frac{2B_i\,\mathrm{E}\left(X_i \mid \Im_{i-1}\right)\mathrm{E}\left(R_iX_i \mid \Im_{i-1}\right)}{\left(A_i^2 - B_i^2\right)} - \frac{\mathrm{E}\left(R_iX_i \mid \Im_{i-1}\right)^2}{\left(A_i^2 - B_i^2\right)}$$

Using the standard properties of conditional variance and co-variance of random variables the arguments of the second expectation in equation (6) can be expressed as,

$$= \left(1-\frac{B_i^2}{A_i^2}\right)\overline{P}_{i-1}^2 + \left[\operatorname{var}\left(X_i\middle|\Im_{i-1}\right) - \frac{covar^2\left(R_i, X_i\middle|\Im_{i-1}\right)}{A_i^2 - B_i^2}\right]$$

Thus, from equation (6), the second-moment of $\overline{P}_i$ at trading time T_i is,

$$\mathrm{E}\left(\overline{P}_i^2\right) = \left(1 - \mathrm{E}\frac{B_i^2}{A_i^2}\right)\mathrm{E}\left(\overline{P}_{i-1}^2\right) + \mathrm{E}\left[\operatorname{var}\left(X_i\middle|\Im_{i-1}\right) - \frac{covar^2\left(R_i, X_i\middle|\Im_{i-1}\right)}{A_i^2 - B_i^2}\right]$$

$$+\,\mathrm{E}\left[A_i^2\,\overline{S}_{i-1}^2\left(\Delta_i - \frac{B_i\hat{\Delta}_{i-1}^T\hat{\delta}\,S_i + B_i\left(\beta - \mathrm{E}\left(R_i X_i\middle|\Im_{i-1}\right)\right)}{\overline{S}_{i-1}A_i^2}\right)^2\right]$$

with X_i as defined in (3). One can now recursively use equation (3) to compute the second-moment of the hedging error as

$$\mathrm{E}\left(\varepsilon^2\right) = \sum_{i=1}^{n}\left\{\prod_{j=i+1}^{n}\left(1-\frac{B_j^2}{A_j^2}\right)\right\}\mathrm{E}\left[A_i^2\,\overline{S}_{i-1}^2\left(\Delta_i - \frac{B_i\hat{\Delta}_{i-1}^T\hat{\delta}\,S_i + B_i\left(\beta - \mathrm{E}\left(R_i X_i\middle|\Im_{i-1}\right)\right)}{\overline{S}_{i-1}A_i^2}\right)^2\right]$$

$$+\sum_{i=1}^{n}\left\{\prod_{j=i+1}^{n}\left(1-\frac{B_j^2}{A_j^2}\right)\right\}\mathrm{E}\left[\operatorname{var}\left(X_i\middle|\Im_{i-1}\right) - \frac{covar^2\left(R_i, X_i\middle|\Im_{i-1}\right)}{\operatorname{var}\left(R_i\middle|\Im_{i-1}\right)}\right]$$

$$+\left[\prod_{j=1}^{n}\left(1-\frac{B_j^2}{A_j^2}\right)\right]\left(\beta - X_0\right)^2 \qquad (10)$$

To minimize $\mathrm{E}\left(\overline{P}_i^2\right)$ for a given β we need to choose $\Delta_i = \Delta_i^\beta$, where,

$$\Delta_i^\beta = \frac{1}{\overline{S}_{i-1}A_i^2}\left[B_i\hat{\Delta}_{i-1}^T\hat{\delta}\,S_i + B_i\left(\beta - \mathrm{E}\left(R_i X_i\middle|\Im_{i-1}\right)\right)\right]$$

To minimize over β as well, we need to choose $\beta = X_0$

The term Δ_i^β, $i \in \{1,2,\cdots,n\}$ in equation (2) is the solution to the second-moment-minimizing hedging positions for a given set of hedge times. Note that from equation (3), using recursion, we could write,

$$\mathrm{E}\left(R_i X_i\middle|\Im_{i-1}\right) = \frac{\mathrm{E}\left(\left(\prod_{j=i+1}^{n}\left(A_j^2 - B_j R_j\right)\right)R_i V\middle|\Im_{i-1}\right)}{\prod_{j=i+1}^{n}\left(A_j^2 - B_j^2\right)}, \quad i \in \{1,\cdots,n\}$$

From the computational perspective, the terms A_j^2 and B_j are deterministic. For a general option, the numerator of (12) can be computed using Monte-Carlo simulations. To this end, for a given index i, the underlying asset prices are to be simulated starting from index $i-1$ up to n (i.e. at all the future hedge times), after which the conditional expectation of (12) is evaluated. In principle, the premium β can be chosen in a arbitrary fashion and the computed Δ_i^{β} , $i \in \{1,2,\cdots,n\}$ are those that minimize the second-moment of the hedging error given that β . It is for this reason, we denote the hedging positions in equation (2) with a superscript β . Choosing $\beta = X_0$ would minimize the second-moment of the hedging error over all choices of β as well.

Relationship with Minimum-Variance Hedging

In this section, we elucidate on the relationship between our second-moment minimizing hedging positions (2) and the minimum-variance positions described in (Bhat, S.P., Chellaboina, Kumar, Bhatia, & Prasad, 2009), (Bhat, S.P., Chellaboina, & Bhatia, 2010). The minimum-variance framework of (Bhat, S.P., Chellaboina, Kumar, Bhatia, & Prasad, 2009), (Bhat, S.P., Chellaboina, & Bhatia, 2010) considers computing hedging positions that minimize the variance of the hedging error in the equivalent risk neutral measure. A measure Θ is called an equivalent risk neutral measure of P if Θ and P agree on the null sets of Ω and the discounted asset price process $\left(e^{(-rt)}S_t\right)$ is a martingale under Θ . Taking expectations (under Θ) on both sides of equation (1) and using the martingale property of the discounted asset process (in Θ), we have,

$$\mathrm{E}^{\Theta}\left(\overline{P}\right) = \beta - \mathrm{E}^{\Theta}\left(\overline{V}\right), \qquad (13)$$

If the premium β is equal to $\mathrm{E}^{\Theta}\left(\overline{V}\right)$, then $\mathrm{E}^{\Theta}\left(\overline{P}\right) = 0$. Hence, the variance in Θ measure is given by,

$$\mathrm{var}^{\Theta}\left(\overline{P}\right) = \mathrm{E}^{\Theta}\left(\overline{P}^{\,2}\right) - \mathrm{E}^{\Theta}\left(\overline{P}\right)^2 = \mathrm{E}^{\Theta}\left(\overline{P}^{\,2}\right) \qquad (14)$$

Hence, in the risk-neutral measure Θ , minimizing the variance is same as minimizing the second-moment.

The hedging positions that yield minimum-variance in the risk-neutral case for a general ECC (see (Bhat , S.P., Chellaboina, Kumar, Bhatia, & Prasad, 2009)) is given by,

$$\Delta_i^* = \frac{\mathrm{cov}\,ar^{\Theta}\left(\overline{S}_i, \overline{V}_i \middle| \mathfrak{I}_{i-1}\right)}{\mathrm{var}^{\Theta}\left(\overline{S}_i \middle| \mathfrak{I}_{i-1}\right)}, \qquad i \in \{1, 2, \cdots, n\} \tag{15}$$

where $\overline{V}_i = \mathrm{E}^{\Theta}\left(\overline{V} \middle| \mathfrak{I}_i\right)$. We will now show that (2) reduces to (15) when $\mathrm{P} = \Theta$. Observe that

$$\begin{aligned} B_i &= \mathrm{E}^{\Theta}(R_i) = \mathrm{E}^{\Theta}\left(\frac{\overline{S}_i}{\overline{S}_{i-1}} - 1\right) \\ &= \mathrm{E}^{\Theta}\left(\frac{1}{\overline{S}_{i-1}} \mathrm{E}^{\Theta}\left(\overline{S}_i - \overline{S}_{i-1}\right) \middle| \mathfrak{I}_{i-1}\right) \\ &= 0 \end{aligned} \tag{16}$$

Observe from (3), we write, $X_i = \mathrm{E}^{\Theta}\left(\overline{V} \middle| \mathfrak{I}_{i-1}\right)$, $i \in \{1, 2, \cdots, n\}$. From (2) and (16), we have,

$$
\begin{aligned}
\Delta_i^{\beta} &= \frac{1}{\bar{S}_{i-1} A_i^2}\left[\mathrm{E}^{\Theta}\left(R_i X_i \middle| \Im_{i-1}\right)\right] \\
&= \frac{1}{\bar{S}_{i-1} A_i^2}\mathrm{E}^{\Theta}\left(\left(\frac{\bar{S}_i}{\bar{S}_{i-1}} - 1\right)\mathrm{E}^{\Theta}\left(\bar{V}\middle|\Im_i\right)\middle|\Im_{i-1}\right) \\
&= \frac{1}{\bar{S}_{i-1} A_i^2}\mathrm{E}^{\Theta}\left(\frac{\bar{S}_i\bar{V}_i - \bar{S}_{i-1}\bar{V}_i}{\bar{S}_{i-1}}\middle|\Im_{i-1}\right) \\
&= \left(\frac{\mathrm{E}^{\Theta}\left(\bar{S}_i\bar{V}_i\middle|\Im_{i-1}\right) - \mathrm{E}^{\Theta}\left(\bar{S}_{i-1}\middle|\Im_{i-1}\right)\mathrm{E}^{\Theta}\left(\bar{V}_i\middle|\Im_{i-1}\right)}{\bar{S}^2{}_{i-1} A_i^2}\right) \\
&= \frac{\operatorname{cov} ar^{\Theta}\left(\bar{S}_i, \bar{V}_i\middle|\Im_{i-1}\right)}{\mathrm{E}^{\Theta}\left(\left(\bar{S}_i - \bar{S}_{i-1}\right)^2\middle|\Im_{i-1}\right)} \\
&= \frac{\operatorname{cov} ar^{\Theta}\left(\bar{S}_i, \bar{V}_i\middle|\Im_{i-1}\right)}{\operatorname{var}^{\Theta}\left(\bar{S}_i\middle|\Im_{i-1}\right)} = \Delta_i^{*}
\end{aligned}
$$

This means that the minimum-variance hedging strategy is same as the second-moment-minimizing hedging strategy in the risk-neutral measure.

Specialization to the European Call Option

In this section, we specialize our second-moment hedging strategy (2) to a European call option with strike price K and time-to-maturity T under the assumption that the underlying asset price process (with no dividends) follows a GBM. Let $W = \{W_t : 0 \le t \le T\}$ be a standard Brownian motion on $(\Omega, \Im, \mathrm{P})$. Let r, μ and $\sigma \in \Re^{+}$ with r as the risk-free interest rate, μ as the drift rate of the underlying and σ as the volatility of the underlying. Then, we define the random variable S_t on $(\Omega, \Im, \mathrm{P})$ as a GBM, given by,

$$
S_t = S_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right) \qquad (17)
$$

The differential form of equation (17) can be obtained using Ito's formula as,

$$dS_t = \sigma S_t dW_t + \mu S_t dt \tag{1}$$

Black and Scholes (Black & Scholes, 1973) asserted that such a call option can be priced uniquely at any given time t, as

$$\begin{aligned} f(x,t') &= x\,\Phi\Big(g(x,t') - Ke^{-rt'}\Phi\big(h(x,t')\big)\Big) \\ g(x,t') &= \left[\ln\left(x/K\right) + \left(r + \frac{1}{2}\sigma^2 t'\right)\right] \Big/ \sigma\sqrt{t'} \\ h(x,t') &= g(x,t') - \sigma\sqrt{t'} \end{aligned} \tag{1}$$

where $\Phi(\cdot)$ is the standard normal distribution function, x is the current stock price and t' is the time to expiration. The following theorem specializes our second-moment-minimizing delta formula (2) to the European call option described above.

Theorem 2: *The solution to the second-moment hedging problem for a European*
call option is given by,

$$\Delta_i^{\beta} = -\frac{1}{\bar{S}_{i-1} A_i^2}\left[B_i \hat{\Delta}_{i-1}^T \hat{\delta}\, S_{i-1} + B_i \beta - D_i\right], i \in \{1,2,\cdots,n\},$$

where,

$$D_i = \frac{e^{(\mu - r)(T_i - T_{i-1})}\nu_{i+1,i}\left(e^{\sigma^2 (T_i - T_{i-1})} S_{i-1}\right) - \nu_{i+1,i}\left(S_{i-1}\right)}{\prod_{j=i+1}^{n}\left(A_j^2 - B_j^2\right)}$$

(The function $\nu_{k,i}(\cdot)$, $k \in (i+1,\cdots,n)$ is evaluated, for each i, by the following recursion

$$v_{k,i}\left(S_{i-1}\right)=\left(A_k^2+B_k\right)v_{k+1,i}\left(S_{i-1}\right)-B_k\, e^{(\mu-r)(T_k-T_{k-1})}\, v_{k+1,i}\left(e^{\sigma^2(T_k-T_{k-1})}S_{i-1}\right)$$

$$v_{n+1,i}\left(S_{i-1}\right)=e^{(-r)(T-T_{i-1})}f\left(e^{(\mu-r)(T-T_{i-1})}S_{i-1},\,T-T_{i-1}\right) \quad ($$

where $f(x,t)$ is the Black-Scholes price function as in equation (19). Further, A_k^2 and B_k can also be evaluated in closed form as

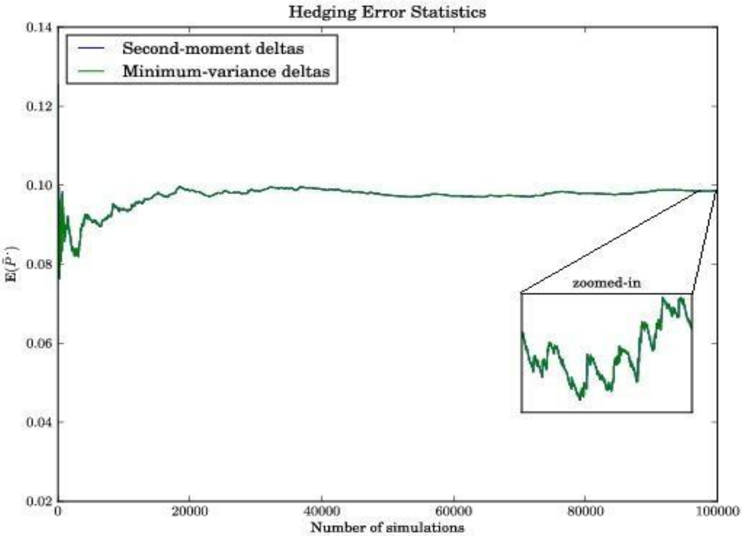
$$A_k^2=e^{2\left(\mu-r+\frac{\sigma^2}{2}\right)(T_k-T_{k-1})}-2e^{(\mu-r)(T_k-T_{k-1})}+1$$

$$B_k=e^{(\mu-r)(T_k-T_{k-1})}-1 \qquad (22)$$

We omit the proof of this theorem in this manuscript due to space constraints. Note that the computation of Δ_i^{β} in the interval $\left(T_{i-1},T_i\right]$ requires the calculation of function $\nu_{k,i}(\cdot)$ in a recursive fashion.

Numerical Simulations

In this section, we present results that illustrate the proposed second-moment hedging paradigm. To this end, we considered a European call option with the underlying as a stock that follows GBM. For the purpose of simulations, the time to maturity of the option (T = 7 days) and the strike was fixed (K = 97). We considered two scenarios, one in which the growth rate of the stock μ was same as the risk-free rate of interest r and another in which $\mu > r$. For each case, we simulated 100,000 stock paths with spot price (S_0 = 100). Hedging was done on a daily basis. The second-moment and the minimum-variance hedging positions were calculated. We used the closed form expressions in equations (20)-(21) to compute the second-moment-minimizing hedging positions described in equation (2). Note that the function $\nu(\cdot)$ in (21) has to be calculated recursively and it involves the computation of the Black-Scholes price in (19). Minimum-variance positions were calculated using equation (15). The hedging positions thus calculated, were used to compute the hedging errors as in equation (1).



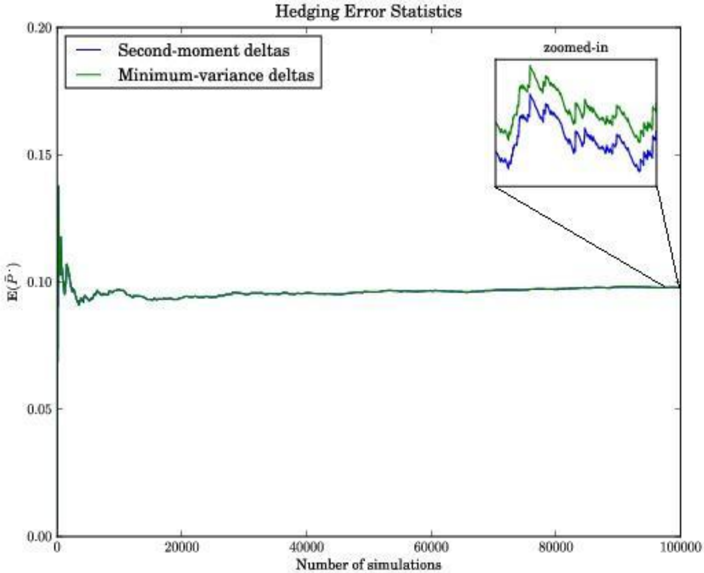


Figure 1(a), 1(b): Plot of the second-moment of the hedging errors versus number of simulations for a European call option with S_0 = 100; K = 97; T = 7 days. Hedging was performed daily (all 7 days).A total of 100,000 paths were considered. The inset contains the plot of last 2000 values of $\mathrm{E}\left(\overline{P}^2\right)$. For the case, $\mu = r$ (Figure 1(a)), the hedging positions are identical [$\mathrm{E}\left(\overline{P}^2\right)$ converge to 0.098599)]. For the case $\mu > r$ (Figure 1(b)), second-moment-minimizing deltas are less 'riskier' in the sense that $\mathrm{E}\left(\overline{P}^2\right)$ converges to a lower value (0.097852) than the minimum-variance framework (0.097930).

Figures 1(a) and 1(b) show the results of our simulations for the two scenarios mentioned above. In both of these figures, we plot the sample second-moment of the hedging error versus the number of simulations. From the inset of the figures, we see that, for the case $\mu = r$, as expected, the second-moment-minimizing hedging strategy is identical to the minimum-variance framework. For the case $\mu > r$, the second-moment-minimizing hedging strategy yields lower second moment for the hedging error than the minimum-variance framework.

Discussion

In our exposition, we proposed that the risk for the seller can be quantified as the second-moment of the final hedging error and subsequently developed a framework to minimize such risk. In the past, other measures for risk had been proposed and hedging strategies were formulated accordingly. Follmer and Schweizer (Follmer & Schweizer, 1989) measured conditional risk in terms of the variance of the incremental cost incurred between successive hedging instants. The work of Schal (Schal, 1994) introduced conditional remaining cost and variance of the hedging errors as measures of risk and argued that, if the discounted asset process is a martingale, the hedging strategy of (Follmer & Schweizer, 1989) reduces these two risks as well. Later Schweizer (Schweizer M. , 1995) proposed a minimum variance strategy for the case where the asset price need not be characterized as a martingale under the assumption that the underlying asset price process has a bounded mean-variance trade-off. The paper of (Schal, 1994) gave solutions to the same problem under the stronger assumption of a deterministic mean-variance trade off. In our paper though, we gave an elementary and self-contained derivation of the hedging strategy that minimizes the second moment of the hedging error under the assumption that the arithmetic returns on the underlying asset across hedging intervals are independent. Our problem is thus a special case of (Schal, 1994) and (Schweizer M. , 1995). However, we go a step further and specialize our results to an ECC written on an underlying whose price process follows a GBM. In this special case, the hedging positions have closed-form formulae in terms of a pricing function for the ECC. For a more general case Monte-Carlo simulations have to be employed.

In a more recent work (Bhat, S.P., Chellaboina, Kumar, Bhatia, & Prasad, 2009), (Bhat, S.P., Chellaboina, & Bhatia, 2010) the authors had suggested a suggested a minimum-variance framework wherein the variance of the hedging error is minimized globally in the risk-neutral measure and provided a computable discrete-time hedging strategy that works well when the stock follows a GBM. An important aspect of the minimum-variance strategy proposed in (Bhat S.P. , Chellaboina, Kumar, Bhatia, & Prasad, 2009), (Bhat, S.P., Chellaboina, & Bhatia, 2010) is that, the risk was quantified as the variance of the final money market position in the risk-neutral measure. As a consequence, the suitability of the hedging strategy to the real world data was limited. The current work, in contrast, considers the second-moment of the hedging errors in the market measure and have many advantages. Firstly, it could very well be applied on the real market data because it makes no assumptions on the underlying measure. Secondly, the framework is same for simple as well as path-dependent claims, differing only in the computation of the payoff. Thirdly, our strategy does not make any assumptions on the model of the underlying asset prices. However, for models that are more complicated than the GBM, the expressions for hedging positions in (2) have to be evaluated using computationally intensive Monte-Carlo techniques. Further, all the hedge times have to be pre-decided to facilitate the implementation of our hedging strategy.

The second-moment-minimizing hedging framework presented here provides the necessary background to develop hedging techniques based on quadratic optimization in the market measure. In a forthcoming work, we use the ideas developed here to analyze a scheme that minimizes the variance of the hedging error in the market measure. The applicability of these hedging schemes to various models and market data and for different kinds of options would also be explored.

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