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Stock Market Modeling in the Langevin Formalism

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Abstract

A Langevin formalism is proposed for stock market dynamics with modeling of various economic market features from first principles. Various processes and effects that occur in the stock market are mathematically incorporated in the said formulation. The Fokker-Planck equation corresponding to the Langevin equation thus obtained is solved. It shows deviation from Gaussian behavior of the rate of change of stock price PDF. The deviation relates to factors such as market efficiency, market depth, liquidity of the relevant stock and informational asymmetries.

Keywords: Stock Market, Langevin Equation, Fokker-Planck Equation, Black-Scholes Model, Market Microstructure

JEL: C020

Introduction

While extensive work has been done in studying stock market behavior using various statistical methodologies, a complete explanation of the stock price patterns on different time scales continues to elude human insight (Guillaume et al., 1997; Mantegna & Stanley, 1995; Cont et al., 1997; Cont, 1997; Cizeau et al., 1997; Potters et al., 1998; Sornette et al., 1996; Feigenbaum & Freund, 1996; Vandewalle et al., 1998). In general, modeling of stock prices follows a standard assumption of their log-normality with the returns being Gaussian (Hull, 1997; Grandmont, 1981; O'Hara, 1995). However, empirical price patterns invariably exhibit deviations from this standard, particularly over small time scales wherein non-Gaussian effects are pronounced. Furthermore, over extended time scales, linear growth of the logarithms of stock prices are observed with rare anomalous events (Guillaume et al., 1997; Mantegna & Stanley, 1995; Cont et al., 1997; Cont, 1997). Several modifications of the standard Black Scholes theory have been attempted to explain these observations, albeit with limited success (Guillaume et al., 1997; Mantegna & Stanley, 1995; Cont et al., 1997; Cont, 1997). A majority of these models look at the stock market at the macroscopic level wherein a dynamic equilibrium is perpetrated through interaction of demand and supply of various securities through the interplay of market agents (Grandmont, 1981; O'Hara, 1995). The micro-structural phenomena that weighs into market trades is not modeled as such (O'Hara, 1995). In this paper, however, a "first principles" approach is adopted. Market trading is modeled by attempting a mathematical representation of the various factors, behavioral aspects, and risk-return influences that contribute to an investor's mindset in entering into a stock market transaction (Markowitz, 1995; Elton & Gruber, 1995). A Langevin type equation is derived incorporating therein several terms that influence market trading undertaken by an investor. The equation is then solved for the probability density function of the rate of change of stock price.

The Black-Scholes Stock Price Model

It is conventional to model the stock price movements over an infinitesimal time interval (t, $t + dt$) by the Ito stochastic differential equation representing a geometric Brownian motion,

$$dS(t) = \mu S(t)dt + \sigma S(t)dW \tag{1}$$

where μ is the expected drift rate (return), σ is the volatility of the stock price at time t and dW is the standard Brownian motion increment over the interval (t, $t + dt$). This increment is normally distributed with mean 0 and variance dt and can, therefore, be expressed as $dW = z\sqrt{dt}$ where z is the standard normal variate. This equation is an Ito process where the coefficients of dt and dW are proportional to the instantaneous stock price S. It is emphasized that this model of stock prices holds over infinitesimal time intervals (t, $t + dt$). The above Ito equation is equivalent to the Langevin equation,

$$\frac{dR(t)}{dt} = \frac{dS(t)/S(t)}{dt} = \mu + \sigma\eta(t) \tag{2}$$

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where $\eta(t)$ is a δ correlated Gaussian (or white) noise with unit variance, viz:

$$\langle \eta(t) \rangle = 0 \tag{3}$$

$$\langle \eta(t)\eta(t') \rangle = \delta(t-t') \tag{4}$$

This model assumes that the instantaneous percentage return $R(t) = dS(t)/S(t)$ on a stock is a function of the drift rate μ and volatility σ of the stock price. The drift rate and volatility are, themselves, constant over the infinitesimal time interval $(t, t + dt)$. Thus, the expected percentage return (μ) required by investors over this infinitesimal interval from a stock is independent of the stock's price. If investors require a certain expected return over $(t, t + dt)$ when the stock price is S_1, then, ceteris paribus, they will also require the same expected return when it is S_2. It follows that the expected drift rate of the stock price over$(t, t + dt)$ is $\mu S(t)$ and the corresponding expected change in stock price over this infinitesimal interval is $\mu S(t)\, dt$. As to the variance of the process, the model assumes that the variability (σ^2) of the percentage return in $(t, t + dt)$ is constant and independent of the stock price, i.e., an investor is just as uncertain of the percentage return when the stock price is S_1 as when it is S_2. It follows that the standard deviation of the change in stock price in $(t, t + dt)$ is $\sigma S(t)$.

The corresponding price process over finite time intervals can be easily obtained by the Ito equation. We have, by setting, $\xi = \ln S$, $dS(t) = \mu S(t)dt + \sigma S(t)dW$ in the Ito equation,

$$d\xi = \left(a\frac{\partial \xi}{\partial x} + \frac{\partial \xi}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 \xi}{\partial x^2} \right)dt + b\frac{\partial \xi}{\partial x}dW$$

for $dx = a(x,t)dt + b(x,t)dW$ that

$$d\xi = \left(\mu - \frac{\sigma^2}{2} \right)dt + \sigma dW \tag{5}$$

whence $d\xi \sim \mathbf{N}\left[\left(\mu - \frac{\sigma^2}{2} \right)dt, \sigma^2 dt \right]$.

Equivalently, $\ln S_T \sim \mathbf{N}\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right)T, \sigma^2 T \right]$

thereby showing that the stock prices follow a lognormal distribution with $\ln S_T$ being normally distributed with mean $\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right)T$ and variance $\sigma^2 T$.

The corresponding Fokker-Planck equation (Appendix) for the transition probability is,

$$\frac{\partial}{\partial t}p(S,t|S',t') = -\frac{\partial}{\partial S}\left[\mu S(t)p(S,t|S',t') \right]$$
$$+\frac{1}{2}\frac{\partial^2}{\partial S^2}\left[\sigma^2 S^2(t)p(S,t|S',t') \right]$$

or

$$\frac{\partial}{\partial t}p = (\sigma^2 - \mu)p + (2\sigma^2 - \mu)S\frac{\partial p}{\partial S} + \frac{1}{2}(\sigma S)^2\frac{\partial^2 p}{\partial S^2} \tag{6}$$

with the boundary conditions,

$$t = t' : p(S,t|S',t') = \delta(S - S') \tag{7}$$

$$S = 0 : p(0,t|S',t') = 0 \tag{8}$$

$$S \to \infty : p(S,t|S',t') \to 0 \tag{9}$$

Justification of the boundary conditions follows from (i) at $t = t'$, stock price $S = S'$, (ii) if the stock price vanishes at any point in time, it stays zero thereafter & vice versa and, on the other hand, if $S(0) > 0$, by assumption, it can never become zero at any later time so that $p(0,t|S',t') = 0$ essentially for $S = 0$ and (iii) the stock price cannot increase unboundedly in a finite time interval. The Fokker-Planck equation has the solution,

$$p(S,t|S',t') = \frac{1}{\sqrt{2\pi(\sigma S)^2(t-t')}} \exp\left\{ -\frac{\left[\ln\left(\frac{S}{S'}\right) - \left(\mu - \frac{\sigma^2}{2}\right)(t-t') \right]^2}{2\sigma^2(t-t')} \right\} \tag{10}$$

In terms of $\xi = \ln S$, the transition probability takes a particularly simple form as,

$$p(\xi,t|\xi',t') = \frac{1}{\sqrt{2\pi\sigma^2(t-t')}} \exp\left\{ -\frac{\left[(\xi - \xi') - \left(\mu - \frac{\sigma^2}{2}\right)(t-t') \right]^2}{2\sigma^2(t-t')} \right\} \tag{11}$$

The Stock Market Modeling from First Principles

Although the Black-Scholes model premises on strong rational assumptions and leads to a particularly attractive and cohesive theory of asset pricing with closed form solutions for several tradeable financial assets including derivatives, the underlying assumption of lognormal distribution of stock prices (normal distribution of returns) over finite time periods does not lend itself particularly well to empirical validation and deviations are ubiquitously observed.

In this work, an attempt is made to arrive at a Langevin type equation from the first principles of stock market dynamics. We make the following assumptions underlying the model:

(a) The price $S(t)$ of a stock at any instant of time t is determined by the demand for the stock $N_d(t)$ and supply into the market of the stock $N_s(t)$ at that instant. Furthermore, the price change of a stock in infinitesimal dt is proportional to $\Delta N = (N_d - N_s)$ so that

$$dS(t) = \nu(\Delta N)dt = \nu(N_d - N_s)dt \tag{12}$$

where ν represents the rate of change of price of the stock corresponding to unit excess demand.

(b) We, now, proceed to construct a phenomenological model for the trading activity viz. $\frac{d(\Delta N)}{dt}$ of a stock. For this purpose, we make the following observations:

- Market activity, i.e., the rate of satisfaction of buy/sell orders varies directly with the number of sell/buy orders existing in the system at any given instant. This is natural because brokers and other intermediaries would execute sell orders with greater motivation if the demand (buy orders) for a stock is large and vice versa. They may, even, execute the orders in the own account.
- Market activity must also include a random component that represents spontaneous buyers and sellers that enter the market occasionally or are tempted by particularly attractive investment opportunities.
- Another important aspect that needs to be incorporated into the modeling is the impact of spontaneous return and risk of the investment opportunities.

Putting all these pieces together, we set up the following model for the trading activity in the stock market:

$$\frac{d(\Delta N)}{dt} = \mu_0 - \theta\Delta N + \alpha(r_t - r_0) - \beta(\sigma_t - \sigma_0) + \frac{\sqrt{2D}}{\nu}\eta(t) \tag{13}$$

The left hand side represents the rate of change of net units of stock in the market which is a typical representation of trading activity. μ_0 is a measure of the threshold rate of market activity while θ is indicative of the frequency of market trades. α and β reflect the marginal contributions of risk and returns to market trading activity. D is a scaling factor that represents the market reaction to new information. $\eta(t)$ is a δ correlated Gaussian (or white) noise with unit variance, viz. $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = \delta(t-t')$.

Combining eqs. (12) & (13), we have a Langevin type equation and obtain

$$\frac{d^2S}{dt^2} - \nu\left[\mu_0 - \theta\Delta N + \alpha(r_t - r_0) - \beta(\sigma_t - \sigma_0)\right] + \sqrt{2D}\eta(t) = 0 \tag{14}$$

Noting that $\Delta N = \frac{1}{\nu}\frac{dS}{dt}$ and $r_t - r_0 \propto \frac{dS}{dt}$ so that $(r_t - r_0) = \varepsilon\frac{dS}{dt}$ we obtain

$$\frac{d^2S}{dt^2} + (\theta - \alpha\nu\varepsilon)\frac{dS}{dt} - \nu\left[\mu_0 - \beta(\sigma_t - \sigma_0)\right] + \sqrt{2D}\eta(t) = 0 \tag{15}$$

Writing $\xi = \frac{dS}{dt}$, we obtain a SDE for the rate of change of stock price as:

$$\frac{d\xi}{dt} + (\theta - \alpha\nu\varepsilon)\xi - \nu\left[\mu_0 - \beta(\sigma_t - \sigma_0)\right] + \sqrt{2D}\eta(t) = 0 \tag{16}$$

or

$$\frac{d\chi}{dt} = -(\theta - \alpha\nu\varepsilon)\chi - (\theta - \alpha\nu\varepsilon)\sqrt{2D}\eta(t) \tag{17}$$

where $\chi = (\theta - \alpha\nu\varepsilon)\xi - \nu\left[\mu_0 - \beta(\sigma_t - \sigma_0)\right]$.

The corresponding Fokker-Planck equation can, now be written:

$$\frac{\partial p(\chi,t)}{\partial t} = (\theta - \alpha\nu\varepsilon)\frac{\partial p(\chi,t)}{\partial\chi} + D(\theta - \alpha\nu\varepsilon)^2\frac{\partial^2 p(\chi,t)}{\partial\chi^2} \tag{18}$$

This equation can be reduced to a diffusion equation by the substitutions:

$$p(\chi,t) = e^{a\chi + bt}q(\chi,t) \tag{19}$$

$$a = -\frac{1}{2D(\theta - \alpha\nu\varepsilon)} \tag{20}$$

$$b = -\frac{1}{4D} \tag{21}$$

whence

$$\frac{\partial q(\chi,t)}{\partial t}=\frac{\partial^2 q(\chi,t)}{\partial \chi^2} \qquad (22)$$

with the Gaussian solution. The complete fundamental solution, then, is

$$p(\chi,t)=\frac{1}{\sqrt{4\pi t}}\exp\left\{-\left[\frac{\chi}{2D(\theta-\alpha\nu\varepsilon)}+\frac{\chi^2}{4t}+\frac{t}{4D}\right]\right\} \qquad (23)$$

Analysis of the Probability Density Function

It is seen from eq. (23) that the probability density function of the rate of change of stock price deviates from the Gaussian behavior under the assumptions of the model. To further explore the implications of the deviant character, we analyze the impact and interpretations of the various parameters that manifest themselves in the PDF.

The parameter θ identifies with the trading activity to net instantaneous market order positions i.e. $\frac{d(\Delta N)/dt}{\Delta N}$. It can, therefore, be related to the liquidity in the market of the particular stock. For liquid stock, θ will be significantly large and we may assume $\theta \gg \alpha\nu\varepsilon$. D represents the conventional diffusion constant. In context of stock market dynamics, it would appear as a measure of the market reaction to random external information. In efficient markets, response to new information would be instantaneous and well distributed indicating a large D. These identification of θ and D would suggest that for highly efficient markets the returns on a highly liquid stock would tend towards the Gaussian distribution in the asymptotic limit. $\nu=\frac{dS(t)/dt}{\Delta N}$ represents the rate of change of stock price corresponding to unit change in excess market demand, i.e., it is, in some sense, a measure of the market's absorbent capacity for the stock. A high ν indicates that a small amount of excess demand can influence the rate of price change significantly and vice versa. Thus, for large efficient markets and stocks with large amount of floating units, it is expected to be small. ε is simply a scaling factor of the measure of stock return.

In eq. (23), the deviation from Gaussian behavior becomes more pronounced with decreasing D indicating market inefficiency with informational asymmetries. These deviations seem to propagate with time.

Conclusions

The Langevin equation (23) has been arrived at on the basis of a first principles economic modeling of many of the processes and effects that take place in a securities market. It needs to be appreciated that a complete capture of market dynamics is impossible with the extant mathematical apparatus and it is, in fact, here that randomness manifests itself. Nevertheless, this first principles model does throw up several interesting features with a modification of the usual Gaussian distribution of rates of stock prices. The deviations that appear from Gaussian behavior are seen to relate to various market features like efficiency, liquidity, trading processes and informational asymmetries.

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Appendix

The Fokker Planck equation for a given Langevin equation

Consider the following Langevin equation:

$$\frac{dx(t)}{dt} = f\left(x(t),t\right) + g\left(x(t),t\right)\eta(t)$$

where $\eta(t)$ is a Gaussian (or white) noise with unit variance, viz:

$$\langle \eta(t) \rangle = 0$$
$$\langle \eta(t)\eta(t') \rangle = \delta(t-t')$$

Discretizing the above equation, we obtain:

$$x(t+dt) - x(t) = f\left(x(t),t\right)dt + g\left(x(t),t\right)\sqrt{dt}\tilde{\eta}(t)$$

where $\tilde{\eta}(t)$ is a random variable with zero mean and unit variance. Let us now consider a generic function $\phi\left(x(t)\right)$. We have,

$$\frac{d}{dt}\phi\left(x(t)\right) = \lim_{dt\to 0}\frac{\phi\left(x(t+dt)\right) - \phi\left(x(t)\right)}{dt}$$
$$= \lim_{dt\to 0}\frac{\phi\left(x(t) + f(x)dt + g(x)\sqrt{dt}\tilde{\eta}\right) - \phi\left(x(t)\right)}{dt}$$
$$\simeq \lim_{dt\to 0}\frac{1}{dt}\left[f(x)\phi'(x)dt + g(x)\tilde{\eta}\phi'(x)\sqrt{dt} + g(x)^2\tilde{\eta}^2\frac{\phi''(x)}{2}dt + o\left((dt)^{3/2}\right)\right]$$

Now, we make the Ito assumption i.e. that when discretizing the Langevin equation we compute $g(x(t))$ at the beginning of the time step, i.e. using the value t, and not $t + dt/2$. It is only if this assumption holds that we can decouple $g(x)\tilde{\eta}$ while taking averages so that $\langle g(x)\tilde{\eta}\rangle = \langle g(x)\rangle\langle\tilde{\eta}\rangle = 0$ since $\langle\tilde{\eta}\rangle = 0$. On the same premise, $\langle g(x)^2\tilde{\eta}^2\rangle = \langle g(x)^2\rangle\langle\tilde{\eta}^2\rangle = \langle g(x)^2\rangle$ so that

$$\left\langle \frac{d\phi\left(x(t)\right)}{dt}\right\rangle = \left\langle f(x)\phi'(x) + g(x)^2\frac{\phi''(x)}{2}\right\rangle$$
$$= \langle f(x)\phi'(x)\rangle + \frac{1}{2}\langle g(x)^2\phi''(x)\rangle$$

In terms of probability distribution $P(x, t)$, we can write the average of any function $F(x(t))$ in the form:

$$\langle F\left(x(t)\right)\rangle = \int dz F(z)P(z,t)$$

so that

$$\left\langle \frac{d\phi\left(x(t)\right)}{dt}\right\rangle = \frac{\partial}{\partial t}\int dz\left[\phi(z)P(z,t)\right]$$
$$\langle f(x)\phi'(x)\rangle = \int dz f(z)\phi'(z)P(z,t)$$
$$\langle g(x)^2\phi''(x)\rangle = \int dz g(z)\phi''(z)P(z,t)$$

Setting $\phi\left(x(t)\right) = \delta\left(x(t) - X\right)$, we obtain

$$\left\langle \frac{d\delta\left(x(t)-X\right)}{dt}\right\rangle = \frac{\partial}{\partial t}\int dz\left[\delta(z-X)P(z,t)\right] = \frac{\partial P(X,t)}{\partial t}$$

$$\langle f(x)\delta'\left(x(t)-X\right)\rangle = \int dz f(z)P(z,t)\frac{d}{dz}\left[\delta(z-X)\right]$$

$$= -\int dz\left\{\frac{d}{dz}\left[f(z)P(z,t)\right]\right\}\left[\delta(z-X)\right] = -\frac{\partial}{\partial X}\left[f(X)P(X,t)\right]$$

$$\langle g(x)^2\delta''\left(x(t)-X\right)\rangle = \int dz g(z)^2 P(z,t)\frac{d^2}{dz^2}\left[\delta(z-X)\right]$$

$$= \int dz\frac{d^2}{dz^2}\left[g(z)^2P(z,t)\right]\left[\delta(z-X)\right]$$

$$= \frac{\partial^2}{\partial X^2}\left[g(X)^2P(X,t)\right]$$

Substituting these expressions in $\left\langle \frac{d\phi\left(x(t)\right)}{dt}\right\rangle = \langle f(x)\phi'(x)\rangle + \frac{1}{2}\langle g(x)^2\phi''(x)\rangle$ we obtain the Fokker Planck equation

$$\frac{\partial P(X,t)}{\partial t} = -\frac{\partial}{\partial X}\left[f(X)P(X,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial X^2}\left[g(X)^2P(X,t)\right]$$