Performance Rating of Transmuted Ailamujia Distribution: An Analytical Approach

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Abstract: In this article, a generalization of the Ailamujia distribution so-called transmuted Ailamujia distribution is proposed and studied. We will use the quadratic rank transmutation map (QRTM) in order to generate a flexible family of probability distributions taking Ailamujia distribution as the base value distribution by introducing a new parameter that would offer more distributional flexibility. We provide a comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior. The superiority of the proposed distribution has been illustrated with an application to a real data set.

Keywords: Ailamujia distribution, Reliability function, Maximum likelihood estimation, Real life application.

I. Introduction

The statistical inferences about various lifetime distributions, such as exponential distribution, Weibull distribution, Erlang distribution, Pareto distribution and normal distribution, etc. have been studied a lot. Recent years, many new distributions are proposed for various engineer application. Ailmujia is one of these distribution proposed by Lv et al. (2002). Pan et al. (2009) studied the interval estimation and hypothesis test of Ailamujia distribution based on small sample. The cumulative distribution function of Ailamujia distribution is given by

$$G(x;\theta,\alpha)=1-(1+2\theta x)e^{-2\theta x} \quad ,x\geq 0,\theta>0 \qquad (1.1)$$

and the probability density function (pdf) corresponding to (1.1) is

$$g(x;\theta,\alpha)=4x\theta^2e^{-2\theta x} \quad ,x\geq 0\ ,\theta>0 \qquad (1.2)$$

From the last few decades researchers are busy to obtain new probability distributions by using different techniques such as compounding, T-X family, discretization, transmutation etc. but transmutation of probability distribution has received maximum attention which is an innovative and sound technique to obtain new probability distributions. The purpose of this study is to present a new generalization of Ailmujia distribution called the transmuted Ailmujia distribution. We will derive the proposed distribution using the Quadratic rank transmutation map proposed by Shaw et al. (2009).

A random variable X is said to have transmuted distribution if its cumulative distribution function is given by

$$F(x)=(1+\lambda)G(x)-\lambda G(x)^2 \quad ,|\lambda|\leq 1 \qquad (1.3)$$

where $G(x)$ is the CDF of the base distribution. If we put $\lambda=0$, we get the base distribution. Afaq et al. (2014) studied transmuted inverse Rayleigh distribution and discussed its properties. Aryal and Tsokos (2011) studied the transmuted extreme distributions. The authors provided the mathematical characterization of transmuted Gumbel and transmuted Weibull distributions and their applications to analyze real data sets. Faton Merovci (2013) studied the transmuted Rayleigh distribution, Ashouret et al (2013) studied the transmuted exponentiated Lomax distribution and discussed some properties of this family. Afaq et al. (2015) proposed transmuted Rayleigh distribution and discussed some of its properties. In the present study we will provide the mathematical formulation of the transmuted Ailmujia distribution and some of its structural properties.

II. Transmuted Ailmujia Distribution

In this section we studied the transmuted Ailmujia (TA) distribution and the sub-models of this distribution. Now using (1.1) in (1.3), we have the CDF of transmuted Ailmujia distribution given by

$$F(x;\theta,\lambda)=\left(\lambda+(1+2\theta x)e^{-2\theta x}\right)\left(1-(1+2\theta x)e^{-2\theta x}\right) \qquad (2.1)$$

Hence the pdf of Transmuted Ailmujia distribution with parameters θ and λ is given as

$$f(x;\theta,\lambda)=4x\theta^2e^{-2\theta x}\left(1-\lambda+2\lambda(1+2\theta x)e^{-2\theta x}\right)$$

If we take $\lambda=0$ in (2.2), we get $f(x;\theta)=4x\theta^2e^{-2\theta x}$ which is pdf of Ailmujia distribution.

III. Reliability Analysis

In this sub-section, we present the reliability function and the hazard function for the proposed transmuted Ailmujia distribution. The reliability function is otherwise known as the survival or survivor function. It is the probability that a system will survive beyond a specified time and it is obtained mathematically as the complement of the cumulative density function (CDF).

The survivor function is given by

$$R(x) = 1 - F(x)$$

$$R(x) = \left(\lambda + (1+2\theta x)e^{-2\theta x}\right)\left(1-(1+2\theta x)e^{-2\theta x}\right)$$

The hazard function is also known as the hazard rate, failure rate, or force of mortality. The hazard rate function is given by

$$h(x) = \frac{f(x)}{1-F(x)} =$$

$$h(x) = \frac{4x\theta^2 e^{-2\theta x}\left(1-\lambda+2\lambda(1+2\theta x)e^{-2\theta x}\right)}{1-\left(\lambda+(1+2\theta x)e^{-2\theta x}\right)\left(1-(1+2\theta x)e^{-2\theta x}\right)}$$

IV. Statistical Properties

In this section we shall discuss structural properties of transmuted Ailmujia distribution. Specially moments, order statistics, maximum likelihood estimation, moment generating function,

Moments:

The following theorem gives the rth moment of the transmuted Ailmujia distribution

Theorem 4.1: If X has the TA(θ,λ) distribution with $|\lambda| \le 1$, then the rth non-central moments are given by

$$\mu_r' = 4\theta^2(1-\lambda)\frac{\Gamma(r+2)}{(2\theta)^{r+2}} + 8\lambda\theta^2\frac{\Gamma(r+2)}{(4\theta)^{r+2}} + 16\lambda\theta^3\frac{\Gamma(r+3)}{(4\theta)^{r+3}}$$

Proof:

$$\mu_r' = \int_0^\infty 4x^{r+1}\theta^2 e^{-2\theta x}\left(1-\lambda+2\lambda(1+2\theta x)e^{-2\theta x}\right)dx$$

$$= 4\theta^2(1-\lambda)\int_0^\infty x^{r+1}e^{-2\theta x}dx + 8\lambda\theta^2\int_0^\infty x^{r+1}e^{-4\theta x}dx + 16\lambda\theta^3 \int_0^\infty x^{r+2}e^{-4\theta x}dx$$

$$\Rightarrow \mu_r' = 4\theta^2(1-\lambda)\frac{\Gamma(r+2)}{(2\theta)^{r+2}} + 8\lambda\theta^2\frac{\Gamma(r+2)}{(4\theta)^{r+2}} + 16\lambda\theta^3\frac{\Gamma(r+3)}{(4\theta)^{r+3}} \quad (4.1)$$

Takingr =1, 2, 3 and 4 in equation (4.1) we get first four moments of transmuted Ailmujia distribution

$$\mu_1' = \frac{8-3\lambda}{8\theta},\ \mu_2' = \frac{24-15\lambda}{16\theta^2},\ \mu_3' = \frac{96-75\lambda}{32\theta^3},$$

$$\mu_4' = \frac{120-105\lambda}{16\theta^4}$$

Also variance $\mu_2 = \dfrac{32-9\lambda^2-12\lambda}{64\theta^2}$

It can be easily seen that for $\lambda = 0$ in equation (4.1) we get rth moment Ailmujia distribution which is given below

$$\mu_r' = 4\theta^2\frac{\Gamma(r+2)}{(2\theta)^{r+2}}$$

and its mean $\mu_1' = \dfrac{1}{\theta}$ and variance $\mu_2 = \dfrac{1}{2\theta^2}$

Moment Generating Function

In this sub section we derived the moment generating function of TA(θ,λ)distribution.

Theorem 4.2: If X has the TA(θ,λ) distribution with $|\lambda| \le 1$, then the moment generating function $M_X(t)$ is given by

$$M_X(t) = \frac{(1-\lambda)}{(2\theta+t)^2} + \frac{8\lambda\theta^2(1+4\theta)}{(4\theta+t)^2}$$

Proof: We begin with the well-known definition of the moment generating function given by

$$= M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x;\theta,\lambda)\,dx$$

$$= 4\theta^2\int_0^\infty e^{-(2\theta+t)x}x\left(1-\lambda+2\lambda(1+2\theta x)e^{-2\theta x}\right)dx$$

$$= 4\theta^2(1-\lambda)\int_0^\infty e^{-(2\theta+t)x}x\,dx + 8\lambda\theta^2\int_0^\infty x(1+2\theta x)e^{-(4\theta+t)x}\,dx$$

$$\Rightarrow M_X(t) = \frac{(1-\lambda)}{(2\theta+t)^2} + \frac{8\lambda\theta^2(1+4\theta)}{(4\theta+t)^2} \quad (4.2)$$

V. Parameter Estimation

In this section the estimation of parameters of $DWBD(x;\gamma,\alpha,\beta)$ model will be discussed through method of moments and maximum likelihood estimation.

A. Moments Method of Estimation

In order estimate two unknown parameters of $TA(x;\theta,\lambda)$ model by the method of moments, we need to equate first two sample moments with their corresponding population moments.

$$m_1 = \gamma_1;\ m_2 = \gamma_2$$

where $\gamma_i = \frac{1}{n}\sum_{i=1}^n x_i$ is the ith sample moment and m_i is the ith corresponding population moment and the solution for $\hat{\theta}$ and

$\hat{\lambda}$ may be obtained by solving above equations simultaneously through numerical methods.

B. Maximum Likelihood Estimation

We estimate the parameters of the transmuted Ailmujia distribution using the method of maximum likelihood estimation (MLE) as follows;

Let $X_1, X_2, ..., X_n$ be a random sample of size n from transmuted Ailmujia distribution. Then the likelihood function is given by

$$L(x\,|\,\alpha,\theta,\lambda)=\prod_{i=1}^{n}\left[4\theta^2 x e^{-2\theta x}\left(1-\lambda+2\lambda(1+2\theta x)e^{-2\theta x}\right)\right] \tag{5.1}$$

By taking logarithm of (5.1), we find the log likelihood function

$$l = n\log 4 + 2n\log\theta + \sum_{i=1}^{n}\log x_i - 2\theta\sum_{i=1}^{n}x_i$$

$$+\sum_{i=1}^{n}\log\left[1-\lambda+2\lambda(1+2\theta x_i)e^{-2\theta x_i}\right]$$

To obtain the MLE's of θ and λ, we differentiating log likelihood with respect to θ and λ

$$\frac{\partial\log L}{\partial\theta}=\frac{2n}{\theta}-2\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{8\lambda\theta x_i^2\, e^{-2\theta x_i}}{(1-\lambda+2\lambda(1+2\theta x_i)e^{-2\theta x_i})} \tag{5.3}$$

$$\frac{\partial\log L}{\partial\lambda}=-\sum_{i=1}^{n}\frac{(1-2(1+2\theta x_i)e^{-2\theta x_i})}{(1-\lambda+2\lambda(1+2\theta x_i)e^{-2\theta x_i})} \tag{5.4}$$

The MLE of θ *and* λ and is obtained by solving these nonlinear system of equations. Setting these expressions to zero and solving them simultaneously yields the maximum likelihood estimates of these three parameters.

VII. Application

The following real data set is considered for illustration of the proposed methodology. The data below are from an accelerated life test of 59 conductors, failure times are in hours, and there are no censored observations Lawless (2003).

2.997, 4.137, 4.288, 4.531, 4.700, 4.706, 5.009, 5.381, 5.434, 5.459, 5.589, 5.640, 5.807, 5.923, 6.033, 6.071, 6.087, 6.129, 6.352, 6.369, 6.476, 6.492, 6.515, 6.522, 6.538, 6.545, 6.573, 6.725, 6.869, 6.923, 6.948, 6.956, 6.958, 7.024, 7.224, 7.365, 7.398, 7.459, 7.489, 7.495, 7.496, 7.543, 7.683, 7.937, 7.945, 7.974, 8.120, 8.336, 8.532, 8.591, 8.687, 8.799, 9.218, 9.254, 9.289, 9.663, 10.092, 10.491, 11.038.

We have fitted Ailmujia and transmuted Ailmujia models to this data. These two distributions are fitted to the subject data using maximum likelihood estimation. The MLEs of the parameters and their corresponding log-likelihood values, AIC, AICC and BIC are displayed in Table 6.1.

Table I: Estimation of Parameters and Comparison Criteria

Distribution	Parameter estimate	Standard error	-2logl	AIC	BIC	AICC
Ailmujia	$\hat{\theta}$=0.1432	0.0131	304.98	306.98	308.52	307.05
Transmuted Ailmujia	$\hat{\theta}$=0.5901 $\hat{\lambda}$ =-0.1417	0.0903 0.3123	99.470	103.47	106.55	103.68

It can be seen from Table 6.1, the Transmuted Ailmujia distribution have the lesser AIC, AICC and BIC values as compared to Ailmujia Distribution. Hence we can conclude that the Transmuted Ailmujia distribution leads to a better fit than the Ailmujia distribution.

VI. Conclusion

In this paper, a two-parameter continuous distribution, namely 'Transmuted Ailmujia (TAD)', of which the Ailmujia distribution (AD) is a particular case, has been proposed. Several properties of the proposed distribution such as moments, survival function, hazard rate function, moment generating function, estimation of parameters by the method of maximum likelihood and the method of moments have been discussed. Finally, an application of the proposed distribution has been given by fitting to data sets relating to failure times of 59 conductors, to test its goodness of fit to which earlier the AD has been fitted and it is found that the Transmuted Ailmujia distribution provides better fits than those by the AD. An application to a real data set indicates that the fit of the proposed distribution is superior to the fit of the Ailmujia distribution and we hope that the proposed distribution may be interesting for a wider range of statistics research.

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