Lagrangian Relaxation for the Capacitated Dynamic Lot Sizing Problem

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Abstract

In this paper, we consider the capacitated dynamic lot-sizing problem and assume that all conditions of Wagner-Whitin (1958) model apply except the capacity restriction. We give three different formulation of the problem. We relax the capacity constraint and initiate the Lagrangian procedure. At each Lagrangian iteration, we solved the uncapacitated lot-sizing problem by the Wagner-Whitin method that runs in $O(n^2)$ time. We compare the quality of bounds so obtained. In particular, we find that the Lagrangian procedure that modified the setup cost turned out to be inferior to the Lagrangian procedure that modified only the holding cost.

Introduction

Production planning is a way of achieving a long term decision related to what work should be done in some interval amount of time or a tentative plan for how much quantity of production should occur in a certain time interval, called planning horizon. It is one of the most challenging issues for a manufacturing industry to have a perfect production plan as it directly relates to effective utilization of resources by improving various parameters including the process flow. It also optimizes the operational cost and improves the timely delivery of a product. In short, it provides improvement in quality and customer satisfaction. However, production planning in itself is a tedious and complex job. A production planner faces various challenges about which product should be produced with how much should be the quantity to be produced keeping all available constraints of production system in mind. Sometimes, the constraints appear to be tightly bound, making it very difficult to achieve an effective production plan. It is an effective utilization of resources to keep the production goals in a certain amount of time. Karimi et al. (2003) broadly describes how the decision making in a production plan can be classified in the time ranges of long-term, medium-term and short-term.

Lot-sizing is a middle-term production planning problem in which decisions related to when and how much quantity to produce over a planning horizon is taken. The objective is to determine the periods in which production happens and the amount of quantity to be produced while utilizing the production resources (minimizing the production, setup and holding cost). Many techniques and solution procedures were proposed to achieve lot-sizes of a lot-sizing problem. As performance of a system and its productivity are the two most important parameters for a manufacturing industry to compete in the market, proper lot-sizing may result in achieving the two. This thus encourages devoted researchers to work for developing and improving the solution procedures of a lot-sizing problem.

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Literature Review

Wagner and Whitin (1958) gave an algorithm to solve a basic single item uncapacitated lot-sizing problem. Zoller and Robrade (1988) made a comparison between various heuristics developed to solve the single-item dynamic lot-sizing problem. Bitran and Yanasse (1982) included the capacity constraint and provided analysis of the problem. Silver & Meal (1973) also gave a heuristic to solve the uncapacitated single item lot sizing problem with a time-varying demand rate. Federgruen & Tzur (1991), Wagelmans et al. (1992) and Aggarwal & Park (1993) reviewed the Single-item uncapacitated lot-sizing problem and present an exact solution procedure. Federgruen & Tzur (1991) provide a forward algorithm to solve the general dynamic lot sizing problem. Wagelmans et al. (1992) gave an O(nlogn) linear time algorithm for the Wagner -Whitin model.

Later for the uncapacitated lot sizing problem case of Wagner–Whitin problem, Aggarwal & Park (1993) provide an algorithm based on dynamic programming. The basic contribution of these works was their attempt to reduce computational complexity as compared to the Wagner–Whitin algorithm.

Karmarkar et al. (1987) studies a single item capacitated lot-sizing problem for both uncapacitated and capacitated cases. Wolsey (1989) referred to this problem as lot sizing with startup costs. Majority of the heuristics for capacitated lot-sizing problem are based on the formulation of Manne (1958).

Practical lot sizing problems are known to be some of the hardest problems to solve. Florian et al. (1980) elaborate complexity results of the single item case of problems, while Chen & Thizy (1990) explain it for multi-item cases. It is shown that the single item capacitated problem lot-sizing problem is NP-hard for quite general objective functions. Capacitated lot sizing problem with concave cost functions (Wagner & Whitin, 2004) are solvable in polynomial time. Lot sizing with convex cost functions and no setup cost is also solvable in polynomial time. Salomon et al. (1991), Vanderbeck (1998) and Webster (1999) later gave some more complexity results of these problems. Because of the difficulty posed by these problems, numerous solution techniques have been explored to solve them.

Thizy & Van Wassenhove (1985) used Lagrangian relaxation and relaxed the capacity constraints to decompose the bigger problem into 'N' sub problems; each of the sub problems are of single item uncapacitated lot sizing. These are solvable by the Wagner–Whitin algorithm. The solution of the Lagrangian problem provides a lower bound, while the upper bound is obtained by first fixing the setup variables given by the dual solution and secondly, by obtaining the solution from the resulting transportation problem.

Lagrangian multipliers are updated using the sub-gradient optimization procedure Held et al. (1974). Lozano et al. (1991) applied the Primal dual approach to solve the Lagrangian relaxation of capacitated lot-sizing problem. Diaby et al. (1992) and Hindi (1995) are some other works using Lagrangian relaxation.

Problem Description

Following are the assumptions:

1. Demand is deterministic and dynamic with planning horizon finite.
2. Per unit production cost is independent of production quantity.
3. Each unit item is produced independent from other units.
4. Lead time is known and is set to zero.
5. Back orders are not allowed.
6. Inventory holding cost is linear and is included at the end of the holding period.
7. Without loss of generality, beginning and ending inventories of a planning horizon are set to zero.
8. Production cost is the same in each period.

Formulation of Problem

The basic uncapacitated dynamic lot-sizing problem is defined as:

$$Z = min\ \Sigma_t f_t * y_t + \Sigma_t h_t * I_t \qquad (1)$$

Constraints:

$$I_{t-1} + X_t = d_t + I_t \qquad \forall t \qquad (2)$$

$$X_t \leq M * y_t \qquad \forall t \qquad (3)$$

$$y_t \in (0,1) \qquad \forall t \qquad (4)$$

$$X_t,\ I_t \geq 0 \qquad \forall t \qquad (5)$$

$$I_0,\ I_T = 0 \qquad (6)$$

We consider two capacity constraints leading to three different scenarios of additional capacity restriction on w-w lot sizing rule:

$$X_t \leq C_t \qquad \forall t \qquad (7)$$

$$X_t \leq C_t * f_t \qquad \forall t \qquad (8)$$

Formulation F1 = minimize (1), Subject to (2), (3), (4), (5), (6) and (7)

Formulation F2 = minimize (1), Subject to (2), (3), (4), (5),(6) and (8)

Formulation F3 = minimize (1), Subject to (2), (3), (4), (5), (6), (7) and (8)

where,

X_t: Quantity of item produced in period t
C_t: Capacity available in period t
y_t: Setup Variable in period t
d_t: Available demand in period t
I_t: Ending inventory in period t
h_t: Per unit Inventory holding cost in period t
T: Total Number of Periods in a planning Horizon
f_t: Fixed setup cost incurred in period t
M: Large Number
λ_t: Lagrangian Multiplier
T: Total number of periods

The Lagrangian Procedure in Brief

Lagrangian relaxation is well appropriate where constraints can be sub-divided into two categories:

- "Easy" constraints: those with which the entire problem can be solved easily.
- "Difficult" constraints: those which make the problem very hard to solve.

The overall idea is to relax the problem. This is done by get ridding of the "Difficult" constraints, by removing them and putting them into the objective function with some assigned weight (the Lagrangian multiplier). Each weight represents a penalty to the solution that does not satisfy the particular constraint.

We are given the following integer linear problem:

$$Z = min\ C^T * \chi$$
$$A * \chi \geq b$$
$$D * \chi \geq d$$
$$\chi \text{ integer}$$

Where,

A, D, b, c, d are all having integer entries

Let X be a set := {x integral | D * x ≥ d}

It is assumed that optimization over *X* can be done easily, whereas after the addition of the "difficult" constraints $A * x \geq b$ makes the problem intractable. Therefore, we introduce a dual variable for every constraint of $A * x \geq b$ and λ (≥ 0) is the vector of dual variables (Lagrangian multipliers) having the same dimension as that of vector b. For a fixed λ (≥ 0), consider the *relaxed problem*

$$Z(\lambda) = min\ C^T * \chi + \lambda^T * (b - A * ?)$$
$$D * \chi \geq d$$
$$\chi \text{ integer}$$

This reduced problem is now solvable to with Wagner-Whitin Algorithm with fixed values of Lagrangian Multiplier. With the mentioned assumptions, it can efficiently compute the optimal value for the relaxed problem with a fixed vector λ.

Thus, in Formulation F1, F2 and F3 after applying Lagrangian relaxation by relaxing the capacity constraints (5), (6) and (5) & (6) respectively. The reduced formulations follow:

Objective function:

$$F1: Z_{lp1} = min\ \Sigma_t f_t * S_t + \Sigma_t \lambda_t * (X_t - C_t) + \Sigma_t h_t * I_t \qquad (9)$$

$$F2: Z_{lp2} = min\ \Sigma_t f_t * S_t + \Sigma_t \lambda_t * (X_t - C_t * f_t) + \Sigma_t h_t * I_t \qquad (10)$$

$$F3: Z_{lp3} = min\ \Sigma_t f_t * S_t + \Sigma_t \lambda_{t1} * (X_t - C_t) + \Sigma_t \lambda_{t2} * (X_t - C_t * Ft) + \Sigma_t h_t * I_t \qquad (11)$$

Each subjected to constraint (2), (3), (4), (5) and (8) respectively.

It is widely known that for every value of *λ > 0, the* Z_{lp} will give lower bound to the respective problem. Thus, maximizing $\Sigma_{i>0} Z_{lp}$ will give a better lower bound. For each feasible solution of Z_{lp} there exists a solution to the main problem which provides the upper bound. The method to improve the value of λ is through adaptive sub-gradient optimization. Fisher (1985) and Fisher (2004) explain the steps of sub-gradient optimization technique and elaborate information related to the quality of bounds.

Empirical Investigation

By varying the product demand, setup cost relative to available capacity, we made 125 different problem data sets; 25 X50, 50X75 and 50X100 respectively.

Results & Discussions

We compare each formulation and its significant difference over the other two by standard t-tests with a 95% confidence interval. The following are the findings:

- Formulation 1 significantly differs with formulation 2 and formulation 3 in number of iteration steps, duality gap and computation time respectively.
- Formulation 2 significantly differs from formulation 3 only in computation time.

Test of Superiority

Out of 125 problem sets, we sort the problem set according to the least duality gap among all three formulation for each problems. In conclusion:

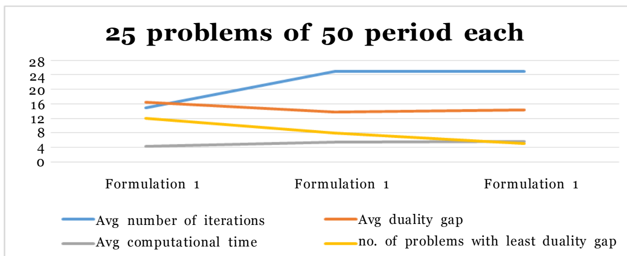

Figure 1: Findings of 25 Problems with 50 Period Each

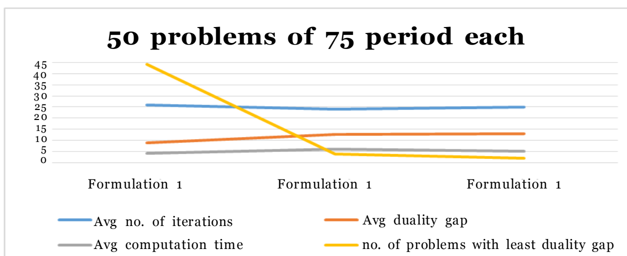

Figure 2: Findings of 50 Problems with 75 Period Each

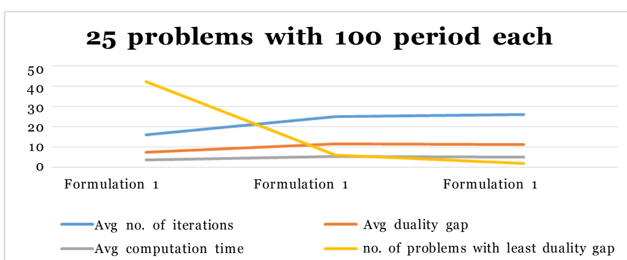

Figure 3: Findings of 50 Problems with 100 Period Each

- For 98 problem set, Formulation 1 appears to be superior.
- For 18 problem set, Formulation 2 appears to be superior.
- For 9 problem set, Formulation 3 appears to be superior.

The sole purpose of sorting is to find whether a significant difference in level of superiority exists or not. We defined hypothesis and made standard t-tests for each of the above sets to know how much one formulation is superior over the other two.

- For 98 problem set in which Formulation 1 is superior, there exists significant difference in number of iteration steps and computation time from both formulation 2 and formulation 3 respectively.

- For 18 problem set in which Formulation 2 is superior, there exists significant difference between formulation 2 and formulation 1 in number of iteration steps, duality gap and computation time. Comparison of formulation 2 with formulation 3 results in significant difference in number of iteration steps and computational time.
- For 9 problem set in which Formulation 3 is superior, there exist significant differences between formulation 3 and formulation 1 in number of iteration. However, when compared to formulation 2, it does not yield any significant difference under the confidence interval of 95%.

Conclusion

We gave three different formulations for single item capacitated lot-sizing problem. We relaxed the capacity constraint using Lagrangian multiplier. We solved the relaxed problem through adaptive sub-gradient optimization technique to achieve lower bound. After varying the product demand, setup cost relative to available capacity, we formed 125 different data sets. The overall finding is that Formulation 1 appears superior in 98 problems. It also gives the tightest bound in less number of iterations among the three. Also, the output is achieved in less computational time. Moreover, Formulation 2 is superior in 18 problem instances and formulation 3 is superior in 9 problem instances. This paper intends to provide information for future research.

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