A Deterministic Description of Irrational and Semi-Rational Bubbles in Asset Markets

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Abstract

This present work provides a deterministic description irrational and semi-rational bubbles based on the stylised description forwarded by Hyman Minsky (1972), Day and Huang (1990) respectively. The paper emphasises on two areas: First, it proposes a mathematical representation of an irrational bubble using piece-wise linear maps in a discrete time frame. Second, it studies the chaotic signals generated by them to explain the instability in asset price bubbles and explains the factors which impact their longevity.

Keywords: Irrational Bubbles, Semi-Rational Bubbles, Piece-Wise Linear Maps

Introduction

One of the most intriguing theoretical problems in the dynamic analysis of asset prices is to understand the causes and forces that determine the nature of fluctuating behaviour typically observed in asset markets. A part of the fluctuations are generated by exogeneous unobservable factors that can be modelled as a stochastic process. Empirical observations as well as theoretical insights into the nature of the asset price movements, indicate that trader's expectation about the future development of asset prices have a significant influence on their actual determination.

An asset price bubble is distinguished from any standard asset price fluctuation by a typical feedback pattern (Shiller, 2002). Such a feedback pattern is observed when increase (decrease) in the price of the asset leads to leads to an enhanced (depressed) investor enthusiasm, which in turn creates an auxiliary increase (decrease) in the asset demand, finally generating more price increases (decreases). Fundamentals do not influence the generation of the feedback pattern, rather it is the result of the confidence of the investors about the optimistic prospect of the asset, based on the returns generated by the asset in the recent past.

Confidence arises as a result of the interaction of the expectations of different heterogeneous agents. Confidence (or panic) depends on the way in which agents change their expectations- that is, the updating rule which they use. Blanchard and Watson (1982), Diba and Grossman (1987), Shiller (1981) tried to explain the expectation formation of agents in the Rational Expectation framework. They also held that expectation distortions could also occur in rational markets. Confidence (or panic) arise as a result of the 'seeming tendency for self-fulfilling rumours about the potential stock price fluctuations' (West, 1987). A vital condition for an asset bubble to occur in a Rational Expectation framework is:

$$E_t (B_{t+1}) = (1 + R)B_t \qquad (1)$$

Where is the expectation conditioned by the information at time't', is the bubble term in the t-th period and R is a constant risk-free rate. (1) Implies that if a bubble is present in an asset price, then any rational investor who is willing to buy the asset must expect it to grow at the rate R. Given , an investor will be willing to buy an asset with inflated price, since he anticipates that he would be compensated through further price rises in the future. Keynes (1936), Samuelson (1957) suggested that rational speculative bubbles were a meagre theoretical likelihood. Investors investing in speculative assets often do not form their expectations too rationally, but, in reality, they tend to herd. The crowding tendencies occur as a result of bounded rationality, or 'less-than-perfect-rationality'.

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Rosser (1997) explained that models with the presence of heterogeneous investors can explain the process of bubble formation much better than those models which assume agents to be equally rational or irrational. Literature documents the heterogeneity of information prevalent among agents in a variety of ways. Some earlier models (Goodman (1968), Zeeman (1974)) classify the investors into 'fundamentalists' who trade based on fundamentals of the asset, and 'chartist' who chase price trends. Other models talk of 'un-informed investors', 'price-informed investors' and 'supply-informed investors' (Gennotte and Leland, 1990). The 'un-informed' and 'price-informed' investors carry the similar meaning to 'Chartists' and 'fundamentalists' respectively. The 'supply-informed' investors are the investor who has information on the future price of the asset. Another type of heterogeneity occurs, which can be explained in terms of 'insiders' and 'outsiders'. In the model 'insiders' are rational speculators who start buying when the asset price is below the fundamental value, and start selling them before the price exceeds the fundamental value. The 'outsiders' buy when the price of the asset starts rising above the fundamental value, and also continue to do so even when the asset become over-valued. The interactions of the heterogeneous expectations generate non-linearity in the asset price.

This work proposes a deterministic description of an irrational bubble. A deterministic description tries to explain the occurrence of an event based on the endogenous factors. In reference to the financial bubbles, the objective is to model the stochastic behaviour underlying the asset price bubbles. The asset price bubble is deterministic, in the sense that, only factors endogenous to the determination of asset prices are considered, in explaining the stochasticity in asset prices.

Description of an Irrational Bubble

Rosser (1994) provided a general framework for the asset price bubble from its formation to burst. It is proposed that a bubble can be distinctly divided into four steps from the formation to burst. The steps are summarised below:

First, an initial displacement of the asset price occurs from its fundamental value. This is usually an upward movement where the displacements are brought about by the investor behaviour or human error. Second, speculation develops as buyers start making money and many such buyers follow suit. Prices rise and 'euphoria' emerges. In the third stage, there is a presence of cognitive bias in taking investment decision. A Cognitive Bias refers to a pattern of deviation which occurs in particular situations which may lead to perceptual distortions, inaccurate judgement. This mainly arises as a result of bounded rationality- a limited capacity of information processing by the agents. Finally, the euphoria is followed by a period of distress where expectations undergo major change. 'Bad News' follows and the bubble crashes, and the stock returns to the fundamental value. The change is discreet in nature.

Minsky (1972), Kindleberger (1989), have explained the occurrence of bubbles as a manifestation of the irrational mob psychology. Deterministic models of asset price bubbles explain the irrationality of mob psychology as an interaction of investors with heterogeneous expectation. This interaction generates non-linearity in the asset price dynamics, which is sometimes observed as a switch over from a bull market to a bear market and vice-versa. Such sudden switches indicate building or bursting bubble.

Deterministic Framework for Irrational Bubbles

Irrational bubbles stem from a more general class of asset price bubbles- Semi-Rational bubbles. These models of asset price bubbles contain both well-informed agents who trade assets based on their fundamental value called 'fundamentalist' and ill-informed agents who chase price trends- 'noise traders' (Goodman (1968), Day and Huang (1990)). These two types of agents are defined as:

Rational Speculators: These traders use the available information in taking their trading decision.

Noise Traders: These traders believe that information collection is expensive, and they follow the trends. If prices are going up, they believe that this would continue and they can make a profit out of it by selling it in the future. This type of investors believes that price rises would last longer than other investor does (Camerer (1989)).

Bubbles arise when the impact of the noise traders exceed the impact of the rationalists for sufficiently long period of time. As the bubble proceeds, the rational speculators gradually start selling off their holdings which are brought by the noise traders. When the first few rational tradersstart selling off, the noise traders get tempted to buy the asset. The observed high returns from the asset in the recent past, retained in the minds of the noise traders explain their huge asset purchases. As larger number of rationalists' starts selling off their holdings, some of the noise traders start coming to terms with the reality and start selling off. These noise traders are those from the lot who already have made a profit and do not want to hold the asset any further. But, there are fresh ones who enter the market in hope earning their share of profit from the asset. These fresh investors are Camerer (1989)'s 'greater fool' as they believe that price rises would last longer, than any other investor does, and are likely to get stuck with the overpriced asset once an excess supply is created. These fresh investors keep up the excess demand and the 'Euphoria' of the asset for some more time, but not eternally.

As the rationalisttraders start bailing-off rapidly given their information on the increasing discrepancy between the asset price and actual value of the asset, the prices peak up and then start plummeting. Some noise traders hold on to the asset, but aggregate expectations undergo a downward drift. Heavy selling by noise traders creates a panic and more traders start selling off in a hurry. The asset price falls sharply, and the increased demand generated from the price increases also stop. This is the point where the bubble crashes and the asset price starts converging to its fundamental value.

Deterministic Representation of an Irrational Bubble:

The main objective of this section is to provide a simple explanation of the formation and burst of an irrational bubble in a deterministic framework and study the different factors which explain its longevity. The representation of the model is based on the following assumptions:

First, the asset has a negligible but positive fundamental value. The asset is issued to raise capital to finance an idea which is expected to have tremendous potential in the future, as popularised by news and print media. This assumption will help us to easily distinguish the 'exaggerated beliefs' from the 'fundamental beliefs'. Since the fundamental beliefs are negligible, therefore any movements in the asset price will be the outcome of 'exaggerated beliefs' of the investors. The 'exaggerated beliefs' are assumed to be the outcome of 'news media', 'groupthink' exercises etc. which induce investors to believe that the belief of their peers about the future returns of the asset are correct.

Second, there are two types of investors- Rational speculators and Noise traders. Rational speculators use all the available information in determining the fundamental value of the asset. It is assumed that the rational traders are of two types. Type 1 rational traders are those who start selling off their assets once it exceeds the fundamental price. Type2 rational traders are those who have some kind of idea about the maximum sustainable price of the asset and hold on to the asset until it nears the peak.For simplicity, it is assumed that the rational speculators do not trade shares in-between them. They either sell their holdings to noise traders.

On the other hand, the noise traders observe the action of the rationalist traders. These traders believe that information accumulation is expensive and hence concentrate only on what everyone else is doing. They believe that what others do reflects the information that the others have and they don't. This act of trying to use the information contained in the decisions of other, makes each person's decision less responsive to their information set. The noise traders do not generate fresh information for others.

Finally, there are no exogeneous shocks in this model. Any stochasticity occurring is endogenous.

The price of the asset, at t = 0 (the first time it is issued) is assumed to be a very small positive quantity, ε (> 0).

$$P(0) = \varepsilon \tag{2}$$

To define the price for this description, the actual asset prices are normalised by dividing with the maximum sustainable value for the asset. Our understanding of the maximum sustainable value for the asset is very similar to the definition of 'carrying capacity' used by Verhulst (1598) in his demonstration of logistic equation. The asset prices are ordered in time and each is divided by the maximum value. The prices of the asset in our description will be a set $F = \{P(t): 0 \le P(t) \le 1\}$. The initial value defined in (1) is done in relation to the maximum sustainable asset value.

The asset price in (1) is also the fundamental value of the asset. This is a simplifying assumption to understand the deviation in the asset prices caused by 'exaggerated beliefs'. Since the fundamental value is negligible, therefore increases in it cannot explain huge increases in the asset price. If any such increases are observed then they would be due to irrational behaviour of noise traders. Some of the great bubbles like the Tulip Mania, the South Sea Bubble, and the Dot-Com Bubble have proved this fact. The source of price increases was the expectation about the future potential of the assets involved in these bubbles. By definition, any price increases above would be a bubble.

The rational traders would make a profit by selling the asset at any time period t=1 onwards. From the second assumption, it can be said that a part of the rational traders start selling off their asset holdings and a part of them retains it back only to sell them off as price reaches the maximum sustainable value. Noise traders would enter the market when they see rationalist traders making profits by selling the asset. So, the proportion of the rationalists who sell off their holdings immediately after the price exceeds the fundamental value must be relatively less than the proportion that hold back the asset. The reason behind this condition is to ensure that Noise traders enter the market. If majority of the rationalists start selling off then the possibility of the bubble may be ruled out. Heavy selling by the rationalist in the initial time period may send out a negative signal to the noise traders and they may opt to stay out rather than enter the market.

Noise traders follow the information circulated by the news and the print media while taking their decision to purchase the asset. They make two observations before entering the market: First, they realise that the 'new idea' has huge future potential and hence, the asset has huge returns in the near and distant future and Second, they see their peers (rational speculators) holding the asset. The second observation creates a sense of 'losing out to peers in the future' in the mind of these traders. They feel that their peers have relevant private information about the asset which they do not have and hence it becomes optimal for them to follow their peers.

For the noise traders it is relatively more important to make profits with their peers by investing in the asset, rather than losing by standing out and not purchasing it (given the possibility of a long run loss, if the idea does not live up to expectations). On the other hand, they feel that if the price of the asset falls, all will lose. This tendency to disregard the 'base-rate probabilities' (of an asset being a good asset) and form the decision based on 'observed similarities to previous patterns' of the behaviour of the asset and peers, has been referred to as 'representative heuristics' by Tversky and Kahneman (1974) ,as 'herding tendency' by Banerjee (1992) and as 'less-than-perfectly-rational behaviour' by Shiller (2002).

The noise traders enter the market when they expect the price of the asset to rise. They form their expectation about the future asset price using the information about the observed price of the asset and the displacement of the price from the initial price. The difference between the rational traders and the noise traders is that the former has an idea about the maximum displacement the asset can have from the initial price (which is also the fundamental value) while the latter assume the displacement to continue indefinitely. The higher the displacement, the more they expect it to occur in the next period. Given this behaviour a simple expectation rule can be proposed for the noise traders:

$$E[P\,(t+1)/I_t] = \delta^t(P(t) - P(0), \delta > 1 \qquad (2)$$

δ^t is the responsiveness of the noise traders to displacements in the price of the asset. The coefficient captures the excitement of the noise traders to the increasing displacements of the asset price from its initial value over successive time periods and it increases with time.   is the information set of the noise traders at time t, which is composed of information on the present price of the asset and the historical displacement. The expectation formation rule explains the fact that noise traders believe the asset prices would rise indefinitely.

BlanchardandWatson (1982), Shiller (1978), Taylor (1977), Tirole (1982, 1985) have shown that deviations from the fundamental price occur as a process of self-full-filling expectations. The noise traders start buying the asset in the current period, in the hope of selling it at a higher price in the next period, which acts as a self-fulfilling process- the expected asset price movements in the present period (t) results in actual asset price movements in the following period (t+1). The actual price of the asset in the model is determined by the following rule:

$$P(t+1) = \alpha,\, E[P(t+1)/I_t],\, \alpha \geq 1 \qquad (3)$$

In the equation, α reflects the aggregate responsiveness of the investors to a unit change in the expected price of the asset. One way in which investors are attracted to buy a particular asset is when the actual asset price realised is at least as large as the expected price of the asset. Under such a scenario, increasingly large numbers of investors are eager to put their money into the asset in the hope of earning huge returns in the future. The belief on the future potential of the 'new idea' becomes stronger, so much so that these investors ignore the fundamentals of the asset. Therefore α is a value at least as large as 1. This is a condition which can explain investor's irrationality in a very simple manner.

Substituting (2) into (3) shows an interesting relation between the prices at time t and (t + 1):

$$P(t+1) = \delta.\ \delta^t(P(t) - P(0))$$

$$\frac{(P(\mathrm{t}) - P(0))}{(1/\alpha.\delta^t)}$$

$$\frac{(P(\mathrm{t}) - P(0))}{a} \text{ where } a = (1/\alpha.\delta^t) \text{ and } a\ \varepsilon\,(0,1) \qquad (4)$$

P (t + 1) is determined by its responsiveness to changes in price expectations, by the sensitivity of the price expectations to the current asset priceand the displacement of the asset price from initial time period. Given the values of the parameters, α and δ, ahigher price of the asset in period t would result in a higher displacement from the initial value, and the higher displacement would cause a higher price in period (t+1). This is called the positive feedback mechanism of a bubble.

The positive feedback pattern would continue as long as noise traders would expect the asset price to rise at increasing rates every subsequent period and the rational speculators hold back the asset. In the vicinity of the maximum sustainable value, the rational speculators increase sell-offs in a discreet manner. The positive feedback mechanism would stop as soon as the system reaches its maximum sustainable value. Mathematically, the problem would be to identify the price corresponding to which price in period (t+1) is 1. The asset price accelerates up to the point where it reaches the maximum sustainable value and the crash begins thereon. This happens at the critical point, tipping point or the catastrophe point where a panic is triggered and the asset price goes for a correction. This correction occurs at a critical time period.

$$\frac{(P(t_c) - P(0))}{(1/\alpha.\delta^t)} = 1$$

$$\Rightarrow P(t_c) = P(0) + (1/\alpha.\delta^t c) \qquad (5)$$

Since, the asset price is not driven by the fundamentals of the firm that issued the asset; it can be assumed that the maximum price is independent of the fundamentals. So, without loss of generality it can be said that the maximum sustainable price is determined solely by the excitement of the noise traders. Therefore: corresponding to

$$P(t_c) = (1/\alpha.\delta^t c) \qquad (6)$$

The rationalist traders have bailed off at this point, earning the maximum profit. The asset is now held by the noise traders, who realise that they are holding an over-valued asset. They start selling off their holdings and the asset price starts to plummet. The negative feedback pattern starts from this point and continues until the fundamental price is reached.

For any subsequent price increases in excess of $P(t_c)$, the price of the asset in period (t_c + 1) would be ($\alpha.\delta^t c^{+1}$). In this period the asset is overvalued, since ($\alpha.\delta^t c^{+1}$). As investors are not willing to overpay, their demand for the asset falls. The falling demand for the asset will be reflected through reduced expectation of price rise- the investors will expect the asset price to fall in each period thereon. An increase in the overvaluation of the asset would adversely affect the price expectations, and so: E [P (t+1)/ I_t] = f (1- P (t)). The actual price in the next period would be determined by the aggregate responsiveness of investors to the future price expectations. It can be expected that Rationalists would be mentally prepared for such overvaluation while 'Noise traders' would overreact. At the point of over-valuation there will be no rationalists who are holding the asset, since they have already bailed out before the catastrophe point. Every subsequent period the noise traders would expect the prices to fall, with the rate of fall dependent on the extent of overvaluation and the reason for this is quite clear: The greater is the extent of over valuation the larger is the fall in price as more and more investors want to sell off their assets. So, the responsiveness of the investors to the unit change in the expected price is weighted by the extent of over-valuation, ($\alpha\,\delta^t - 1$). We propose the following function to denote the price of the asset during the downward feedback pattern:

$$P(t+1) = \frac{\alpha.\delta^t\,[1 - P(t)]}{\alpha.\delta^t - 1}$$

$$\Rightarrow P(t+1) = \frac{[1 - P(t)]}{\left[1 - \dfrac{1}{\alpha.\delta^t}\right]}$$

$$\Rightarrow P(t+1) = \frac{1 - P(\mathrm{t})}{1 - a}\ a \le P(\mathrm{t}) \le 1 \qquad (7)$$

The irrational bubble from build to burst has been summarized using the equation:

$$P(t+1) = \frac{P(t)}{a} \quad a \le P(t) \le a$$

$$= \frac{1-P(t)}{1-a} \quad a < P(t) \le 1, a \in (0,1) \tag{8}$$

The set of equations in (8) are piece-wise linear maps, also called 'tent maps'. These maps generate turbulent dynamics within its deterministic structure.

Turbulence in a Deterministic Bubble

The deterministic structure of a bubble has been used to study the role of the endogenous factors in determining the formation of the bubbles. The endogenous disturbances in asset prices arise from the heterogeneity in the expectation formation of the agents. The interaction of the rational speculators and the noise traders create turbulence, even in the absence of the exogeneous shocks, which can be explained by studying the price dynamics (Day and Huang (1990); De Grauwe, Dewatcher & Embredts (1993)).

The endogenous stochasticity of a deterministic system can be explained in a chaos-theoretic framework. A chaotic system is one where the transition of the system from one state to another can be given a probabilistic description despite the system involving no exogenous random variables. Chaotic systems are characterised by trajectories that locally diverge away from one another, and are sensitive to both initial value and parameters. Some general characteristics of a chaotic system are:

(i) The map of a chaotic dynamic system has a non-degenerate invariant measure, μ. The invariant measure, μ, is said to exist for a map '*f*' if: $\mu(f^{-1}(S)) = \mu(S)$ for all measurable sets S. This set is called the 'Attractor Set' for the entire set.

(ii) A small error in measuring the initial conditions, even with the error being infinitesimally small, grows exponentially fast, as't' grows in the short term, until one knows nothing. The idea underlying this assumption is much significant. It requires that the agents must have the potential ability, to measure any initial condition (x_0,say) with infinite accuracy, to forecast x(t) perfectly.

(iii) A very basic property of deterministic chaos is that the time-series {x (t)} appears stochastic even though it is generated by the deterministic system.

Deterministic models of asset price bubbles explain bubbles as non-linear movements arising from the interaction of heterogeneous agents. Bubbles are identified as discreet regime changes from bulls to bear regimes (Day & Huang(1990); De Long, Schlefier, Summers and Waldman (1990); Leland (1990) etc.).

Existence of bubbles in asset prices are identified through sharp and substantive breaks in their time series trend, completely undetermined by what went before. The time series generated by the set of spectrum of the tent-maps and its auto- covariance function are the same as that generated by i.i.d uniform random variables. This equation generates the same correlation coefficients as the first order autoregressive process:

$P(t + 1) = (k.a - 1). P(t) + u(t + 1)$, where k is a constant (9)

The AR (1) process generates a random series for. $a = \frac{1}{k}$ can take positive real values greater than 1. For a < 1/k the system generates negative correlation coefficient, while for a > 1/k the system generates positive correlation coefficients. An important characteristic of the set of difference equations defined in (8) is its behaviour at a = ½. Here the system is symmetric in nature. For values of a < ½, or for a > ½ the rates of rises and fall in the asset prices differ. The two possibilities can be considered here: (i) $f'(P(t))_{P(t)<a} < f'(P(t))_{P(t)>a}$ (ii) $f'(P(t))_{P(t)<a} > f'(P(t))_{P(t)>a}$

Consider the First Possibility

$$f'(P(t))_{P(t)<a} < f'(P(t))_{P(t)>a}$$

$$\Rightarrow \frac{1}{a} < \frac{1}{1-a},$$

$$\Rightarrow a > \frac{1}{2} \tag{10}$$

If a > ½ the fall in the prices is steeper than the rise in the asset prices, i.e. the bubble would crash in a span of time which is much shorter than the time than it would take to build. Similarly, if a < ½ the rise in prices are much faster than the fall in prices. Another observation about the irrational bubbles is: The longevity of an irrational bubble is negatively related to the responsiveness of the market to a unit change in the price of the asset.

A very high value of implies a very low value of 'a'. This means, the high responsiveness of the market to price changes, cause asset prices to reach its peak faster and hence the bubble bursts more quickly.

Price path of an asset during a bubble is characterized by swings, which generally converge to its fundamental value over time. Despite an initial displacement from its fundamental value, the asset price converges to the latter over a finite period of time. The average displacement in the price of the asset from its fundamental value over a sufficiently long period of time is zero. The average displacement of the asset price is defined as:

A (t) = (Total displacement in asset price by time't') / (Total time period)

= P(t) – P(0)/t

Suppose now, there exists a price for the asset at whichthe investors expect the price of the asset to settle down, at a price X* (say), such that P (t) = P (t+1) = P (t+2) =…..=P (t + k) = X*. Over t periods, there would be a sequence of average displacements: {A (t)} = {X*/1, X*/2, X*/3… X*/T}   t = 1 to T When T grows sufficiently large, we can say: {A (t)} →0 as T→∞. If equilibrium to the asset price exists, it would be obtained along the region of diminishing displacement, i.e. along the downward feedback pattern. Zero displacement is the attractor in the set of displacements since trajectories of the tent map all converge to a zero average displacement. For any prices P (t) > a, the asset is over-valued and investors are reluctant to pay for it. The demand for the asset starts falling and so does the price. Not all investors realize the over-valuation and the buying continues, though the selling activities dominate. Every increase in the over-valuation of the asset is followed by a diminishing displacement of the asset price from its fundamental price. Finally, when P (t) = 1, all investors know that the asset is over-valued and they start selling off in a rush. The demand for the asset, and hence its price falls to zero. Therefore, the asset price is attracted to P (t) =1, corresponding to which the average displacement is zero.

Conclusion

The deterministic theories of asset price bubbles are a real good alternative to the standard theories of rational bubbles. The deterministic theories of asset price bubbles explain the occurrence and the burst based on the interaction of the heterogenous expectation of different kinds of investors. Critical results relating to the occurrence and the burst of financial bubbles can be established from this representation. This is a quite simplified framework for representing bubbles in a deterministic framework, and is constructed under certain simplifying assumptions. Further complicated models can be developed under more generalised scenarios.

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