A Supply Chain Collaboration Model for Product Development with R&D Subsidies

Manoj Kumar*, Jyoti Raman*, Priya*

**International Engineering Services, India. E-mail: kumarm1968@rediffmail.com*

Abstract

The capability to concurrently design the product and the supply chain is becoming a key competence in manufacturing companies. In spite of this development, this competence is still underdeveloped in industry. Research has not been able to fill this industrial capability gap partly because there is a lack of convergence of the methodologies for concurrent product and supply chain design in the research community. Today, businesses depend on strategic collaboration with their suppliers and customers to create value to develop product and to obtain better market-share. Designing products to match the processes and supply chains processes to match product platforms and supply chains, and supply chains to match the product platforms and process are the ingredients in today's fast developing markets. If this co-design is done well upfront with sufficient focus on product development process managing, product will cost much less overall and the time to market will decrease substantially. This paper presents a supply chain collaboration dynamic model with two innovative R&D sectors for each supplier and buyer: a vertical R&D sector that improves the quality of existing differentiated products and a horizontal R&D sector that creates new differentiated products. The supplier and buyer exchange differentiated products and beneficiate from knowledge spillovers (possibly impulsed by R&D subsidies). The long term policy effects of R&D subsidies in this context had been studied in this paper. In this contribution, we have realised an attempt to integrate the product development model in a supply chain collaboration framework. This enables us to discuss of the optimal research policy integrating some feedback effects from innovation and knowledge spillovers. Our main result is that the effect of a subsidy to vertical R&D (the only subsidy that has a long term effect) depends on the relative innovative capacities of the supplier or buyer that realised this policy.

Keywords: Product Development Process, Supply Chain Collaboration, Product Innovation, R&D Subsidies

Introduction

For a manufacturing company to be successful in complex and globalised world, the capability to design product has to be complemented by the capability to manage a complex supply chain that delivers the product to the market. However, researchers in product design (PD) and supply chain (SC) management have kept mainly within their domains for various reasons such as complexity of cross-disciplinary research or simply due to unexhausted mono-disciplinary research potentials (Zhang *et al.*, 2008, Ellram *et al.*, 2007). Nowadays, firms should be able to launch into the market a growing number of new products. However, there is no reason why a supply chain that is optimal for a given set of product lines stays optimal when the level of variety changes. Therefore, each time a new product is launched in the market, the supply chain should be redesigned so to be able to deliver the new product efficiently and effectively to the market. Product features affect supply chain performance, while being defined during New Product Development (NPD). The magnitude of the effects of product features on supply chain performance is determined by supply chain decisions concerning supply chain structure (Blackhurts *et al.*, 2005), supply chain strategy, e.g. agile or lean (Childerhouse *et al.*, 2002), or the degree of collaboration among the actors of the supply chain (Doran *et al.*,2007). Within this context aligning supply chain design decisions with new product development decisions has become crucially important to maintain high supply chain performance and to boost product launch effectiveness (Van Hoek & Chapman, 2006; Fine, 1998; Lee & Sasser,1995). However, despite the complex interdependencies among product design and supply chain design decisions have been recognized as early as Hoekstra & Romme (1992), until Fine (1995) this insight did not enter the realms of competitive strategy nor capture the attention of top management (Forza & Rungtusanatham, 2005). Even though many researchers have already identified the benefits of concurrent design of products and supply chains such as greater supply chain performance and risk mitigating flexibility as well a slower supply chain costs, few have systematically quantified these benefits by using complex industrial cases (Blackhurts *et al.*, 2005; Ellram & Stanley, 2008; ElMaraghy & Mahmoudi, 2009; Gokhan *et al.*, 2010; Lu *et al.*, 2007). This paper aims to bridge the gap between the two distinct but equally important research domains.

This paper is introduced in the context of a study on the relations between the supply chain collaboration and product design. The importance of beginning the study of supply chains collaboration in product development process is mainly because it is at this product of lifecycle phase that the decisions responsible for 80% (eighty percent) of a product's final costs are made (Fredrik *et al.*, 2002). The objective of this paper is to build a framework enabling to measure the effects of knowledge spillovers and technical progress direction on growth in supply chain collaboration. In order to complete this objective, we propose a supplier and buyer two-R& D sectors model. The supplier and buyer have innovative capacities and exchange differentiated products that can be introduced and whose qualities can be upgraded through R& D activities. There are two directions for technical progress and thus the research and development efforts and innovation process can be divided in two innovative R& D sectors in each supplier and buyer. The vertical R& D sector that improves the quality level of an existing differentiated products. The horizontal R& D sector introduces new differentiated products. The growth rate of the number of varieties and of the average quality of differentiated products can differ between the supplier and buyer because they have different capacities in terms of horizontal and vertical R& D. Depending on the technical progress direction it has privilegiated and the differentiated products gains world market shares. In a supply chain, supplier's and buyer's innovation policy (in our model a subsidy to vertical or horizontal R& D) can have different long term effects on growth. The objective of this paper is thus to investigate the implication of an innovation policy on the supply chain collaboration.

This paper is organised as follows. In the next section we review some relevant papers on this topic that have different contributions. Models are developed in section 3 with equilibrium conditions. We specify buyer's product utility function, supplier's and buyer's demand function of differentiated products, and supplier and buyer's profit functions. The system equations for both product innovations sectors of supplier and buyer will be developed in this section as well. It is developed the model for an innovation values of supplier and buyer. In section 4, we examine the equilibrium determination, dynamical system of the model; varieties growth rates of products, level for relative average qualities and number of varieties and buyer's product utility growth rate. In section 5, we determine the optimal research policy in supply chain collaboration. A numerical example simulation will illustrate subsidy either the vertical R& D sector, either the horizontal R& D sector or both sectors indifferently and results are discussed. The conclusion is in section 6. All detailed calculations are summarised in Appendix.

Literature Review

The development of new products and improvement of existing products, are considered to be critical to the survival of a supply chain facing tough competition in globalisation. Due to an increase in competition and rapid advancement of technology, innovation and new product development are becoming the essential strategies for the management of the suppliers to survive in the world (Peterson *et al*., 2003). Product innovation is an effective tool that a supplier can utilize to maintain its competitive position in the market. The position of the supplier in the market can be influenced by the frequency of releasing new or improved products (Pisano, 1997). Product dynamism refers to the continuous change in the product which is characterised by speed and magnitude of technological change in the product. To cope up with competition and dynamic change in market demand many suppliers are adopting open innovation which involves collaboration with external entities, like customers, suppliers, universities, research organisations and competitors (Vonhippel, 1988). Many suppliers are switching towards open innovation models, from traditional closed innovation models, to increase their level of innovation and retain competitive position in the supply chain. Open innovation involves interaction of suppliers with several external entities such as customers and suppliers to generate new ideas to improve existing products and develop new products. Not as the previous study shows, both supplier and buyer have positive relationship with innovation and imitation rates. Of all the different types of external sources of collaboration adopted for innovation of the product, suppliers' innovation has the most significant impact on the performance and success of the supply chain (Park & Oduntan, 2010). Supplier collaboration increases the technology knowledge available to a supply chain, which helps it reduce the product development time by identifying the potential problems beforehand (Kessler *et al.*, 1996). Ragatz *et al.* (2002) found that involvement of suppliers in innovation process result in benefits such cost reduction, improved quality and sales increase. The successful development of supplier collaboration involves shared training, mutual trust and commitment, rewards and agreed performance measures (Ragatz *et al.*, 2002). Peterson *et al.* (2003) argued that the integration of suppliers into new product development activities can reduce the risk involved with technology which is at its formative stage. Buyers which share the "technology road-maps" with the suppliers can help the supply chain to introduce first product with new technology which is considered as a competitive advantage (Peterson *et al.*, 2003). Supplier innovation reduces potential problems during the early stages of development (Wasti *et al.*, 1999). Although significant benefits can be achieved through the use of supplier collaboration, for a company seeking to leverage the innovation of its supplier through collaboration, it is necessary that the correct attributes are considered in the design of supplier collaboration relationships. In their paper, these are supposed to be a quality in the product, it will be improved level by level and will be charged with quality adjust price. The model is built upon their previous work on product development and featured innovation and technology transfer. The probability for successful innovation or imitate is random but the aggregate (or average) rate is constant. The equilibrium is then characterised by constant aggregate rates of innovation and imitation. Even this paper is regarded as the benchmark in this area. The objective of our research is to gain a better understanding of the factors that a supply chain must consider in its design of a supplier collaboration relationship.

An analysis of the literature shows that the main product characteristics that affect the relations among product features, supply chain features and client's needs are: product variety and product structure, i.e. product architecture and bill of materials, and product innovativeness. Table 1 summarises the supply chain related variables, along with performance impacted by product features. The analysis of the literature also explores a framework for analysing the state-of-the-art in research with a focus on design trade-off attributes and methodologies used in concurrent product and supply chain design (CP-SCD). Table 2 summarises the design trade-off attributes and methodologies used.

Legend:

Trade-off design attributes (Architectural (A), Detailed (D), Dynamic (Y))
Trade-off methodologies (Mixed-Integer Linear Programming (MILP), Weighted Goal-Programming (WGP), Genetic Algorithm (GA), Non-dominated Sorting Genetic Algorithm (NSGA-II))

Table 1: Literature Analysis

Product Feature(s)	Supply Chain Related Variable(s)	Performance	Reference(s)
Bill of Material	Supply chain structure	Total cost of supply chain	Huang, Zhang & Liang, 2005; Blackhurts, Wu & O'Grady, 2005; Lee & Sasser, 1995
Architecture	Sourcing, postponement strategy, supply chain structure	Costs, service level	Hsuan Mikkola & Skjøtt-Larsen, 2004; Novak & Eppinger, 2001; Hsuan, 2001; Van Hoeck, 2001;Lee& Tang, 1998; Feitzinger & Lee, 1997; Fine, 1995
Variety and architecture	Sourcing, production scales, supply chain configuration	Operational performances	Salvador, Rungtusanatham & Forza, 2004; Salvador, Forza & Rungtusanatham, 2002;
Variety	Manufacturing	Direct costs, overhead, delivery lead times, inventory	Ocampoy Vilas & Vandaele, 2002; Ramdas & Sawhney, 2001; Fisher, Ramdas & Ulrich, 1999; Fisher & Ittner, 1999; MacDuffie, Sethuraman & Fisher, 1996
	Supplier change	Costs	Prasad, 1998
	Information systems	Costs and demand mismatch	Coronado et al. 2004, Jiao et al. 2005, Forza & Salvador 2002
	Manufacturing flexibility	Costs	De Silveira, 1998
Product innovativeness	Supply chain strategy	Operational performance and service level	Fisher, 1997; Childerhouse et al. 2002

Table 2: Literature Analysis of Methodology (in order of publication year)

	Product			Supply Chain			
Authors	**A**	**D**	**Y**	**A**	**D**	**Y**	**Trade-off Method**
Lee & Sasser (1995)	1	1		1	3	1	Optimisation
Krikke et al. (2003)	1		1	1	3		MILP
Blackhurst et al. (2005)		1			2	1	Simulation
Fine et al. (2005)	1			1	5	1	WGP
Huang et al. (2005)	3				4		GA
Su et al. (2005)	2	1			3	1	Simulation
Lamothe et al. (2006)	2		1	1	4		MILP
Zhang et al. (2008)	3	1			3		MILP
Seliger & Zettl (2008)	3	2	2	2	1		MILP
ElMaraghy & Mahmoudi (09)	1				4		MILP
Gokhan et al. (2010)	1		1		3		MILP
El HadjKhalaf et al. (2011)	2		1	2			MILP
Izuiet al. (2010)		2			4	1	Simulation
Jiang et al. (2011)	1				1	1	NSGA-II
Ülkü & Schmidt (2011)	1	2			1	1	Simulation
Nepal et al. (2012)	1				3		WGP
Baud-Lavigne et al. (2012)	1		1		2		MILP
Attribute type count	24	10	6	8	48	6	

Product Variety

Nowadays, customers demand for customized products forces firms to increase product range, i.e. increase the variety they offer in the market. Product variety encompasses both external variety, i.e. the range perceived by the clients, and internal variety, i.e. the diversity of components and semi-finished products (Pil & Holweg, 2004). Product variety is defined during NPD process. This decision affects supply chain performance. For instance, when product variety increases, direct manufacturing costs, manufacturing overhead, delivery times and inventory levels also increase (Ocampo,Vilas & Vandaele, 2002; Ramdas & Sawhney, 2001; Fisher & Ittner, 1999; Fisher *et al.*, 1999; MacDuffie *et al.*, 1996). Brun *et al.* (2006) introduce and define the concept of behavioural costs as "those costs which arise because of the reaction of people to "excessive" variety". In particular, these costs are due to human and/or organisational mechanisms which prevent the available variety to be effectively tackled and deployed. They rise in all those cases when people think the decisional task to choose among various options is not that relevant or could take much time to be completed, so that they exploit less variety than the designed one. To deal with higher variety some tools, e.g. information systems, web-based platforms or flexible automated systems (Coronado *et al.*, 2004; Jiao *et al.*,2005; Forza and Salvador, 2002), should be implemented, thus increasing costs as well (Fisher & Ittner 1999). Prasad proposes a rough index to measure of cost of variety connected to not only manufacturing costs but also plant layout or supplier changes (Prasad, 1998). The magnitude of the impact of variety on the supply chain performance depends on SCM choices. For instance, the impacts of variety on a firm depend on its inherent flexibility (Ramdas, 2003; Berry & Cooper, 1999) and centralisation degree of final assembly (Tynjälä & Eloranta, 2007). De Silveira (1998) develops a framework for the choice of the proper flexibility strategy todeal with high product variety in manufacturing environment. Some empirical and conceptual researches extended this concept to some aspects of SCM (Salvador *et al.*, 2002; Randall & Ulrich, 2001).

Product Structure and Innovativeness

Product design is one of the product-related drivers which impact most SCM decisions and supply chain performance (Salvador *et al.*, 2002). Indeed, product design information is needed for generating manufacturing plans and schedule, and also forcreating a packing plan for shipment (He *et al.*, 2006). Two representations of product design are mainly addressed: product architecture and Bill of Materials. Product architecture plays a pivotal role among NPD and SCM (Krishnan & Ulrich, 2001).

Some relations among product architecture and SCM have been investigated (Fixson, 2005), in particular, focusing on sourcing (Novak & Eppinger, 2001; Hsuan, 2001), postponement strategy and implementation decisions (Hsuan Mikkola & Skjøtt-Larsen, 2004; Van Hoeck, 2001; Lee & Tang, 1998; Feitzinger & Lee, 1997) and supply chain structure (Salvador *et al.*, 2004; Fine, 1995). Mathematical models that support designers in choosing the best Bill of Materials, or generic Bill of Materials, that minimizes the total cost of the supply chain have been proposed as well (Huang *et al.*, 2005; Blackhurst *et al.*, 2005; Lee & Sasses, 1995). In these models, supply chain structure is defined concurrently with the product, among a set of possible configurations. There is a strong relation among product structure and product variety. Variety is mainly addressed in NPD literature in the main trade-off "variety – commonality", i.e. the architecture definition phase (Ulrich & Eppinger, 2000), or in the platform definition one (Huang *et al.*, 2005; Farrell & Simpson, 2003; Martin & Ishii, 2002; Krishnana & Gupta, 2001; Fisher *et al.*,1999; Robertson & Ulrich, 1998). Product architecture decision affects the commercial variety that can be proposed in the marketplace at a given cost (Ulrich, 1995). As far as innovativeness is concerned, it is the degree of newness of a product. It has been studied mainly inrelation to supply chain strategy definition (Fisher, 1997; Childerhouse *et al.* 2002), although the empirical work by Caridi *et al.*(2009) shows the impact of product innovativeness on supply chain operative choices too.

Design Trade-off Methodology

The search for global design optimality of both the product and the supply chain requires methodologies to support trade-off decisions between conflicting design objectives. Trade-off methodology is a pivotal methodology for concurrent product and supply chain design (CP-SCD). Trade-off methodology can be defined as an analytical approach for evaluating and comparing competing design solutions based on stakeholder-defined criteria (Bahill *et al.*, 2001). For design trade-off, Multi-Criteria Decision Analysis (MCDA) methodologies are particularly relevant. Colson & Bruyn (1989) classify Multi-Criteria Decision Analysis (MCDA) into compensatory, non-compensatory or partially compensatory types. For compensatory type, the value of one criterion can be used to compensate the performance of the other (i.e. a trade-off is possible). This requires criteria to becom mensurable. For non-compensatory types, trade-offis not possible due to lack of direct commensurability. Guitouni & Martel (1998) state the need for aggregation of criteria in decision trade-off. Aggregation allows compensation between different criteria and hence enables trade-off to occur. In the context of our review, we define trade-off methodology as the process of finding the best overall solution (global solution) to a problem based on a set of target objectives, evaluation criteria and constraints using commensuration, compensation and aggregation. Multi-Criteria Decision Analysis (MCDA) methodologies that are of particular interest to trade-off are those of compensatory and partially compensatory types such as Weighted-Sum, *Multi-attribute utility theory* (MAUT), Elimination and Choice Expressing Reality (ELECTRE), Analytic Hierarchy Process (AHP) and Multi-Objective Programming (MOP). Detailed descriptions of the algorithms and a comparison between the methodologies can be found in (Guitouni & Martel, 1998; Ehrgott *et al.*, 2010). Simulation is another type of methodology that can be used to support trade-off analysis. Simulation is not a trade-off methodology per sebut can be used with other methodologies (e.g. MCDA, Design of Experiment) to analyse more complex trade-off (e.g. over time) and with stochastic model attributes (Izui *et al.*, 2010; Su *et al.*, 2005).

Gaps in Literature in Supply Chain Collaboration for Product Development

(a) The supply chain contracts can be a useful mechanism to resolve the conflict and risk related problems in product development. The use of information technology in handling transactions online between supply chain members reduces the response time for new product development. The members can plan their operational activities by sharing or retrieving the data from each other. It helps in streamlining the processes and reduces product development costs.

(b) The members might have different technologies, skill and different type of knowledge about their products. To handle any future exceptions or uncertainties in product development, the members may jointly plan supply chain activities like ordering, replenishment, and forecasting and product design.

(c) The following gaps regarding collaboration mechanisms need attention to enhance collaboration for product development:

- Since the role and utility of all collaboration mechanisms for product development is handling different phases of supply chain. To collaborate supply chain as a whole, the consideration of all collaboration mechanisms may give very good performance for product development.
- Most of the models describing collaboration mechanisms are dealt in two level supply chain, which can be extended to multi-level supply chain for product development. The relation between different collaboration mechanisms and the performance measures of supply chain need to be developed. The models handling the problems of collaboration have emphasized on single performance measures. The supply chain dynamics may be captured by considering a number of performance measures of supply chain.
- Supply chain contracts are designed to motivate the downstream member to order more than his/her optimal order quantity. The downstream member always faces uncertainty of overstock or under stock. The upstream member always faces uncertainty that whether the downstream member will send the order matching the upstream member's capacity. The contracts like buyback and revenue sharing contracts can enhance expected sales and reduces stock outs. Quantity flexibility contracts can reduce the overstock problems of downstream members. These performance indicators are equally important, which needs more research attention.
- The contract decision variables at different interfaces of the supply chain in multi echelon environment interact with each other for product development. For example the contract adopted for product development by supplier and manufacturer is sometimes dependent on the contract adopted by same manufacturer with his/her distributor in a same supply chain. There is a need to explore such relationship and to explore different combinations of contracts at different interfaces of supply chain.

(d) Most of the studies are restricted to two level serial supply chains. In reality, supply chain can have divergent and convergent multi-echelon structures. The literature seems lacking to address the uncertainty concerns in such structures for product development.

(b) The literature has emphasised more on demand uncertainty, whereas, supply uncertainty can be of equal concern in the era of globalisation and outsourcing.

Moreover, the quantitative models for product development can be proposed to explore the impact of supply uncertainty on supply chain performance.

(c) There are very few studies on splitting the single period order into multiple orders. The supply chain members can take advantage of more accurate information over a period of selling season and hence resolve product development inefficiencies.

(d) The buyback contract is the only contract which has been discussed in multi-ordering models to manage the risk. There is a scope to explore combination of other contracts for product development in multiple ordering over single season.

Concerning the evaluation of papers on supply chain collaboration with product development, reports on how to involve them with product development are rare. In addition, many of these practices are already in use after the product design phases, during production. There is a lack of papers that deal with the implementation of supply chain collaboration with product development, of methods and tools to aid in this involvement. Literature focuses a lot on reporting what is being done, hardly approaching how to do it.

The Model

Assumption

(i) The quality increment is identical across buyer or supplier inside this but differs between the buyer and the supplier.

(ii) The quality level associated with a new variety of products created at time in supplier or buyer is linked to the average level of weighted qualities at time .

(iii) The diffusion channels of part of the knowledge spillovers comes of reverse engineering for supplier or buyer so that the level of the share for supplier will be function of the level and the level of the sharefor buyer will be function of the level.

Notations:

ρ : the subjective discount rate

σ : the constant elasticity of substitution between product across buyer and supplier

θ : the fraction of consumption expenditures devoted to differentiated product produced by

supplier S

$(1-\theta)$: the fraction of consumption expenditures devoted to differentiated product produced by buyer B

$d_B(t)$: the per capita consumption expenditures at time in buyer B

$d_S(t)$: the per capita consumption expenditures at time in supplier S

$L_B(t)$: the population size of buyer or global supply of labour at time in buyer B

$L_S(t)$: the population size of supplier or global supply of labour at time in supplier S

$Q_B(t)$: the average level of weighted qualities in buyer at time t

$Q_S(t)$: the average level of weighted qualities in supplier at time t

β_B^h : the productivity in horizontal research and product development in buyer B

β_S^v : the productivity in vertical research and product development in buyer B

β_S^h : the productivity in horizontal research and product development in supplier S

β_S^v : the productivity in vertical research and product development in supplier S

$K_S(t)$: an inter-temporal knowledge spillovers of buyer if $k_B \neq 0$

$K_S(t)$: an inter-temporal knowledge spillovers of supplier if $k_s \neq 0$

$L_B^{RH}(t)$: the labour devoted to horizontal R& D in buyer at time t

$L_S^{RH}(t)$: the labour devoted to horizontal R& D in supplier at time t

$L_B^{Rv}(t)$: the labour devoted to vertical R& D in buyer at time t

$L_B^{Rv}(t)$: the labour devoted to vertical R& D in supplier at time t

$L_B^{Rv}(t)$: the labour wage rate at time in supplier S

$L_S^{Rv}(t)$: the labour wage rate at time t in buyer B

$w_s(t)$: total consumption expenditures at time t in supplier S

$w_B(t)$: total consumption expenditures at time t in buyer B

$D_s(t)$: total consumption expenditures at time t in supply chain (including both supplier and buyer)

$D_B(t)$: the flow of profits for producing the differentiated products in buyer attime t

$D(t)$: the flow of profits for producing the differentiated products in supplier attime t

$\pi_B(z, t)$: the average level of weighted quality of the differentiated products in buyer attime t

$\pi_S(u, t)$: the average level of weighted quality of the differentiated products in supplier attime t

$q^B(z, t)$: the lowest adjusted price of the differentiated products in buyer attime t

$q^S(u, t)$: the lowest adjusted price of the differentiated products in supplier attime t

$p^B(z, t)$: the vertical innovation value of products in buyer attime t

$p^S(u, t)$: the vertical innovation value of products in supplier attime t

$V_B^v(z, t)$: the horizontal innovation value of products in buyer attime t

$V_S^v(u, t)$: the horizontal innovation value of products in supplier attime t

$V_B^h(z, t)$: the rate of possibility of horizontal research subsidy in buyer B

$V_S^h(u, t)$: the rate of possibility of horizontal research subsidy in supplier S

r_B^h : the rate of possibility of vertical research subsidy in buyer B

r_S^h : the rate of possibility of vertical research subsidy in supplier S

τ_B^v : the fraction of researcher devoted to vertical R& D in buyer B

τ_S^v : the fraction of researcher devoted to vertical R& D in supplier S

$v_B(t)$: the fraction of researcher devoted to horizontal R& D in supplier S

$v_S(t)$: the fraction of researcher devoted to horizontal R& D in buyer B

$h_S(t)$: the number of quality improvements that have occurred of products in buyer attime t

$h_B(t)$: the number of quality improvements that have occurred of products in supplier at time t

$m(z, t)$: quality increment associated with a vertical innovation in buyer B

$m(u, t)$: quality increment associated with a vertical innovation in supplier S

λ_B : the number of new differentiated products created and produced in buyer at time t

λ_S : the number of new differentiated products created and produced in supplier at time t

$N_B(t)$: the probability to upgrade the quality of products z in buyer attime t

$N_S(t)$: the probability to upgrade the quality of products u in supplier attime t

$\phi_B(z, t)$: the apparent productivity parameter perceived by researchers in buyer B

$\phi_S(z, t)$: the apparent productivity parameter perceived by researchers in supplier S

Buyer's and Supplier's Profit

The labour wage rate, quality of product and total consumption expenditures are used to calculate the profits of buyer and supplier. One unit of labour is required for producing one unit of output whatever its producer either buyer or supplier and the differentiated product quality level. The marginal costs are equal to the wage rates in each of the buyer and supplier ($W_B(t)$ and $W_S(t)$). The total consumption expenditures in buyer is $D_B(t) = d_B(t) L_B(t)$. The total consumption expenditures in supplier is $D_S(t) = d_S(t) L_S(t)$. The total consumption expenditures in supply chain (including both buyer and supplier) is $D(t) = D_B(t) + D_S(t)$. Thus the flow of profits $\pi_B(z, t)$ for producing thedifferentiated products z attime t in buyer B, with quality $q^B(z, t)$ is given by:

$$\pi_B(z,t) = \frac{(p^B(z,t) - w_B(t))\,(q^B(z,t))^{\sigma-1}(p^B(z,t))^{-\sigma}\,(1-\theta)D(t)}{\int_o^{N_B}(q^B(z,t))^{\sigma-1}(p^B(z,t))^{1-\sigma}\,dz} \tag{1}$$

and the flow of profits $\pi_S(u, t)$ for producing the differentiated products u in supplier S at time t with quality $q^s(u,t)$ is given by:

$$\pi_S(u,t) = \frac{(p^S(u,t) - w_S(t))\,(q^S(u,t))^{\sigma-1}(p^S(u,t))^{-\sigma}\,(\theta)D(t)}{\int_o^{N_S}(q^S(u,t))^{\sigma-1}(p^S(z,t))^{1-\sigma}\,du} \tag{2}$$

Integrating prices in the profits functions enables to express the flow of profits of innovative sectors in both buyer and supplier as follows:

$$\pi_B(z,t) = \frac{(p^B(z,t)^{\sigma-1}\,(1-\theta)\,D(t)}{\sigma N_B(t)\,Q_B(t)} \tag{3}$$

Where, $Q_B(t) = N_B(t)-1\int_0^{NB}(z,t))^{\sigma-1}dz$, the average level of weighted qualities in buyer at time t.

$$\pi_S(u,t) = \frac{(q^s(u,t))^{s-1}\,(\theta)\,D(t)}{\sigma N_S(t) Q_S(t)} \tag{4}$$

where, $Q_S(t) = N_S(t)^{-1}\int_0^{N_S}(q^s(u,t))^{\sigma-1}$, the average level of weighted qualities in supplier S at time.

Product Innovation Sectors of Supplier and Buyer

Both supplier and buyer have innovative capacities either in vertical R& D sector and horizontal R& D sector, that signifies that both supplier and buyer enlarges the continuum of the products it produces and ameliorates the quality of the existing products.

Product Horizontal Innovation Sectors of Supplier and Buyer

The flow of new differentiated products that appear in supplier at time is given by:

$$N_s(t) = \beta_s^h (K_s(t))^\gamma L_S^{RH}(t) \tag{5}$$

with $K_S(t) = (N_S(t))^{1-ks}(N_B(t))^{ks}$

where, $0 \ll \gamma \ll 1$ and, $0 \ll k_s \ll 1$.

The productivity parameter of supplier depends on four variables or effects:

- a constant parameter β_S^h reflecting the productivity of researcher in this research activity.
- The researcher devoted to horizontal innovation R&D sectors in supplier S at time t, $L_S^{RH}(t)$.
- The number of new differentiated products created and produced in supplier S at time t, $N_S(t)$.
- The number of new differentiated products created and produced in buyer B at time t, $N_B(t)$.

We assume that the quality level associated with a new variety of products created in supplier at time is linked to the average level of weighted qualities.

- $Q_S(t) = N_S(t)^{-1} \int_0^{N_S} (q^S(u,t))^{\sigma-1} du$
- We postulate that the quality of a horizontal innovation arriving at time t is
- $q^S(t) = Q_S(t)^{\frac{1}{\sigma-1}}$

Symmetrically, the flow of new differentiated products that appear in buyer B at time t is given by:

$$N_B(t) = \beta_S^h (K_B(t))n\, L_B^{RH}(t) \quad (6)$$

with $K_B(t) = (N_B(t))^{1-\kappa_B} (\mathrm{NS}(t))^{\kappa_B}$

where, $0 \ll \eta \ll 1$ and, $0 \ll \kappa_B \ll 1$.

The productivity parameter of buyer depends on four variables or effects:

- a constant parameter β_S^h reflecting the productivity of researcher in this research activity.
- the researcher devoted to horizontal innovation R&D sectors in buyer B at time, $L_B^{RH}(t)$.
- The number of new differentiated products created and produced in buyer B at time t, $N_B(t)$.
- The number of new differentiated products created and produced in supplier S at time t, $N_S(t)$.

Similarly, we assume that the quality level associated with a new variety of products created in buyer at time is linked to the average level of weighted qualities.

$$Q_B(t) = N_B(t)^{-1} \int_0^{N_B} (q^B(z,t))^{\sigma-1} dz$$

We postulate that the quality of a horizontal innovation arriving at time is

$$q^B(t) = Q_B(t)^{\frac{1}{\sigma-1}} \quad (7)$$

In order to have a feedback effect of the market shares of innovative products or innovative processes, we introduce the following assumptions on the diffusion channels of the knowledge spillovers. Part of the knowledge spillovers comes of reverse engineering so that the level of the share for buyer will be function of the level of. For example, if is rather high, the buyer has an advantage in terms of market shares because it has better innovative capacities, either in horizontal or vertical research sector. Thus buyer has little to learn from suppliers technologies. It is thus more beneficiates from its buyer knowledge spillovers than from the supplier knowledge spillovers.

Thus, we can express the constant κ_B as follows:

$$\kappa_B = \theta * \varepsilon_B \quad (8)$$

where, ε_B is constant positive parameter.

Similarly, we can express the constant κ_S as follows:

$$\kappa_S = \theta * \varepsilon_S \quad (9)$$

where, ε_S is constant positive parameter.

Product Vertical Innovation Sectors of Supplier and Buyer

The probability $\phi_S(u,t)$ to upgrade the quality of product in supplier S at time t is given by:

$$\emptyset_S(u,t) = \tilde{\beta}_S^v(u,t) L_S^{Rv}(u,t) \quad (10)$$

with $\tilde{\beta}_S^v(u,t) = \beta_S^v \frac{Q_S(t)}{\left(q^S(u,t)\right)^{\sigma-1}} \left(K_S(t)\right)^{\gamma\mu} \left(\frac{L_S^{Rv}(u,t)}{N_S(t)}\right)^{\mu-1}$

where $\tilde{\beta}_S^v(u,t)$ is the apparent productivity parameter perceived by researchers in supplier S of the continuum, and $L_B^{RH}(u,t)$ is the researcher devoted to the vertical research activity in this sector of the differentiated products continuum in supplier S. The productivity parameter depends on the five variables or effects:

- a constant parameter β_S^v reflecting the productivity of researcher in this research activity,
- a positive research externality coming from the current level of qualities in the supplier S $Q_S(t) = N_S(t)^{-1} \int_0^{N_S} (q^S(u,t))^{\sigma-1} du$,
- a positive research externality coming from the horizontal research sector $(K_S(t))^{\gamma\mu}$,

- an index of research difficulty given by the current sector level of quality at the moment of the innovation race $((q^S(u,t))^{\sigma-1})$,
- a negative externality due to research duplication $(\frac{L_S^{Rv}(u,t)}{N_S(t)})^{\mu-1}$ (with $O < \mu < 1$) when researchers undertake parallel or similar projects in the same sector. Remember that t $N_S(t)$ is the number of differentiated products (technologies) ever created in supplier S attime t, products on which vertical research can be targeted.

Symmetrically (with $0 < \delta < 1$), the probability $\phi_B(Z, t)$ to upgrade the quality of product z in buyer B at time t is given by: to upgrade the quality of product

$$\emptyset_B(z,t) = \tilde{\beta}_B^v(z,t)L_B^{Rv}(z,t) \quad (11)$$

with = $\tilde{\beta}_B^v(z,t) = \beta_B^v \frac{Q_B(t)}{(q^B(z,t))^{\sigma-1}}(K_B(t))^{\eta\delta}(\frac{L_B^{Rv}(z,t)}{N_B(t)})^{\delta-1}$

where $\tilde{\beta}_B^v(z,t)$ is the apparent productivity parameter perceived by researchers in buyer B of the continuum, and $L_B^{Rv}(z,t)$ is the researcher devoted to the vertical research activity in this sector of the differentiated products continuum in buyer B. The productivity parameter depends on the five variables or effects:

- a constant parameter β_B^v reflecting the productivity of researcher in this research activity,
- a positive research externality coming from the current level of qualities in the buyer B $Q_B(t) = N_B(t)^{-1}\int_0^{N_B}(q^B(z,t))^{\sigma-1}dz$,
- a positive research externality coming from the horizontal research sector $(K_B(t))^{\eta\delta}$,
- an index of research difficulty given by the current sector level of quality at the moment of the innovation race $((q^B(z,t))^{\sigma-1})$,,

a negative externality due to research duplication $(\frac{L_B^{Rv}(z,t)}{N_B(t)})^{\delta-1}$ with $0 < \delta < 1$) when researchers undertake parallel or similar projects in the same sector. Remember that t $N_B(t)$ is the number of differentiated products (technologies) ever created in buyer B at time t, products on which vertical research can be targeted.

Product Innovation Values for Supplier and Buyer

There is free entry in research sectors so that equilibrium horizontal innovation values for supplier and buyer are as follows:

$$V_S^h(u,t) = \frac{w_S(t)(1-r_S^h)}{\beta_S^h(K_S(t))^{\gamma}} \quad (12)$$

$$V_B^h(z,t) = \frac{w_B(t)(1-r_B^h)}{\beta_B^h(K_B(t))^{\eta}}$$

The horizontal innovation values are obtained in supplier S (given the possibility of horizontal research subsidy at rate r_S^h) from the horizontal research maximisation problem

$$\left(\max_{L_S^{Rh}(t)} \dot{N}_S(t)V_S^h(t) - w_S(t)(1-r_S^h)L_S^{Rh}(t)\right)$$

Symmetrically, the horizontal innovation values are obtained in buyer B (given the possibility of horizontal research subsidy at rate r_S^h) from the horizontal research maximisation problem

$$\left(\max_{L_B^{Rh}(t)} \dot{N}_B(t)V_B^h(t) - w_B(t)(1-r_B^h)L_B^{Rh}(t)\right)$$

There is free entry in research sectors so that equilibrium vertical innovation values for supplier and buyer are as follows:

$$V_S^v(u,t) = \frac{w_S(t)(1-r_S^v)(q^S(u,t))^{\sigma-1}(L_S^{Rv}(t))^{1-\mu}}{\beta_S^v Q_S(t)(N_S(t))^{1-\mu}(K_S(t))^{\gamma\mu}} \quad (13)$$

$$V_B^v(z,t) = \frac{w_B(t)(1-r_B^v)(q^B(z,t))^{\sigma-1}(L_B^{Rv}(t))^{1-\delta}}{\beta_B^v Q_B(t)(N_B(t))^{1-\delta}(K_B(t))^{\eta\delta}}$$

The vertical innovation values are obtained from the vertical research maximisation problem of research firms in each sector of the continuum of supplier S

$$\left(\max_{L_S^{Rv}(t)} \phi_S(u,t)V_S^v(u,t) - w_S(t)(1-\tau_S^v)L_S^{Rv}(t)\right) \quad (14)$$

where, $\emptyset_S(u,t) = \tilde{\beta}_S^v(u,t)L_S^{Rv}(u,t)$

Whereas, given the possibility of a vertical research subsidy at rate τ_S^v in supplier S.

Symmetrically, the vertical innovation value are obtained from the vertical research maximisation problem research firms in each sector of the continuum of buyer B

$$\left(\max_{L_B^{Rv}(t)} \phi_B(z,t)V_B^v(z,t) - w_B(t)(1-\tau_B^v)L_B^{Rv}(t)\right) \quad (15)$$

where, $\emptyset_B(z,t) = \tilde{\beta}_B^v(z,t)L_B^{Rv}(z,t)$

Whereas, given the possibility of a vertical research subsidy at rate τ_S^v in buyer B.

Product Utility Function of Supplier and Buyer

The supply chain collaboration consists of supplier and buyer. There are given number of consumers in each supplier and buyer. There is a continuum of products indexed by $Z \in [O, N_i]$ in each supplier or buyer (S, B) for producing differentiated products whose qualities can be improved through vertical R& D. N_i is the number of products created and produced in supplier or buyer (S, B). New consumption products can be introduced through horizontal R& D. The quality level of supplier or buyer (S, B) product z is given by $\lambda_i^{m(z,t)}$ where $\lambda_i > 1$ is the quality increment associated with a vertical innovation and $m(z, t)$ is the number of quality improvements that have occurred in productat time t. We assume that the quality increment is identical in products in supplier or buyer but differs between the suppler and buyer. The discounted life time product utility of a consumer in supplier or buyer (S, B) is the following:

$$U_i = \int_0^{\infty} e^{-(\rho - n_i)\tau} \log\big(u_i(\tau)\big) d\tau \qquad (16)$$

Withis $\rho > n_i$ the subjective discount rate of an consumer either of supplier or buyer (S, B). The instantaneous per capita product utility function of a supplier or buyer (S, B) consumer at time is defined by:

$$u_i(t) = \left(\int_0^{N_B} \int_0^{N_B} \left(q^B(z,t)\right)^{\sigma-1} \left(p^B(z,t)\right)^{1-\sigma} dz\right)^{\frac{\sigma(1-\theta)}{\sigma-1}}$$

$$\left(\int_0^{N_S} \left(q^S(u,t)\right)^{\sigma-1} \left(p^S(u,t)\right)^{1-\sigma} du\right)^{\frac{\sigma\theta}{\sigma-1}}$$

Demand Function of Differentiated Products of Supplier and Buyer

Let $x_i(z, t)$ is the per-capita quantity of differentiated product, z produced in supplier or buyer$i$ (S, B), that is consumed by consumers in supplier or buyer (S, B). σ is the constant elasticity of substitution between product across buyer and supplier. θ is the fraction of consumption expenditures devoted to differentiated product produced by supplier S and $(1 - \theta)$ the fraction of consumption expenditures devoted to differentiated product produced by buyer B. Lets $d_i(t)$ be the per-capita consumption expenditures at time in supplier or buyer (S, B) and $p_i(z, t)$ the lowest quality adjusted price for the product produced in supplier or buyer (S, B) at time t. The individual demand function for the lowest quality adjusted product z at time t of the representative consumers in supplier or buyer is given by (Appendix 1):

$$x_B(z,t) = \frac{\left(p_B(z,t)\right)^{-\sigma} \left(q_B(z,t)\right)^{\sigma-1} \left(\int_0^{NB} p_B(z,t) q_B(z,t) dz\right)}{\left(\int_0^{NB} \left(p_B(z,t)\right)^{-\sigma} \left(q_B(z,t)\right)^{\sigma-1} dz\right)} \qquad (17)$$

$$x_S(u,t) = \frac{\left(p_S(u,t)\right)^{-\sigma} \left(q_S(u,t)\right)^{\sigma-1} \left(\int_0^{NS} p_S(u,t) q_S(u,t) du\right)}{\left(\int_0^{NS} \left(p_S(u,t)\right)^{-\sigma} \left(q_S(u,t)\right)^{\sigma-1} du\right)}$$

The total demand is given by,

$$d_i(t) = \int_0^{N_B} \left(q^B(z,t)\right)^{\sigma-1} \left(p^B(z,t)\right)^{1-\sigma} dz + \int_0^{N_S} \left(q^S(u,t)\right)^{\sigma-1} \left(p^S(u,t)\right)^{1-\sigma} du$$

is given by the following expression (Appendix 2):

$$\frac{\int_0^{NS} p^S(u,t) x_i(u,t) du}{\int_0^{NB} p^S(u,t) x_i(u,t) du + \int_0^{NS} p^S(u,t) x_i(u,t) du} \qquad (18)$$

Equilibrium Conditions

There are six equilibrium conditions: two no-arbitrage conditions (one for each innovation sector) in each supplier and buyer and the full employment condition of researchers in each supplier and buyer.

Non Arbitrage Conditions

$$r + \phi_S(u,t) = \frac{\pi_S(u,t)}{V_S^h(u,t)} + \frac{\dot{V}_S^h(u,t)}{V_S^h(u,t)} \qquad (19)$$

$$r + \phi_B(z,t) = \frac{\pi_B(z,t)}{V_B^h(z,t)} + \frac{\dot{V}_B^h(z,t)}{V_B^h(z,t)} \qquad (20)$$

$$r + \phi_S(u,t) = \frac{\pi_S(u,t)}{V_S^v(u,t)} + E\left[\frac{\dot{V}_S^v(u,t)}{V_S^v(u,t)}\right] \qquad (21)$$

$$r + \phi_B(z,t) = \frac{\pi_B(z,t)}{V_B^v(z,t)} + E\left[\frac{\dot{V}_B^v(z,t)}{V_B^v(z,t)}\right] \qquad (22)$$

Full Employment Conditions of Researchers in Supplier and Buyer

To close the model we must describe the researcher markets in both supplier and buyer. In each supplier and buyer, the researchers supply must be equal to the total demand of researchers. The researcher used in manufacturing in supplier is:

$$L_S^X(t) = \int_0^{N_S} L_S^X(u,t) du \qquad (23)$$

Similarly, the researcher used in manufacturing in buyer is:

$$L_B^X(t) = \int_0^{N_B} L_B^X(z,t)dz \tag{24}$$

The number of researcher in horizontal R& D activity is obtained from the equation describing the flow of net varieties at time in supplier :

$$L_S^{Rh}(t) = \frac{\dot{N}_S(t)}{\beta_S^h (K_S(t))^\gamma} \tag{25}$$

The number of researcher in horizontal R& D activity is obtained from the equation describing the flow of net varieties at time in buyer :

$$L_B^{Rh}(t) = \frac{\dot{N}_B(t)}{\beta_B^h (K_B(t))^\eta} \tag{26}$$

In vertical R & D sectors, the number of researchers can be derived using the equation describing probability to innovate in supplier :

$$L_S^{Rv}(t) = \int_0^{N_S} L_S^{Rv}(u,t)du = \int_0^{N_S} \frac{\phi_S(u,t)}{\beta_S^v} du \tag{27}$$

We define $= \frac{L_S^{Rv}(t)}{L_S(t)}$ as the fraction of researcher devoted to vertical R&D in supplier S (constant along the balanced growth path).

In vertical R&D sectors, the number of researchers can be derived using the equation describing probability to innovate in buyer :

$$L_B^{Rv}(t) = \int_0^{N_B} L_B^{Rv}(z,t)dz = \int_0^{N_B} \frac{\phi_B(z,t)}{\beta_B^v} dz \tag{28}$$

We define $v_B(t) = \frac{L_B^{Rv}(t)}{L_B(t)}$ as the fraction of researcher devoted to vertical R & Din buyer B (constant along the balanced growth path).

$$L_S(t) = \int_0^{N_S} L_S^X(u,t)du + L_S^{Rh}(t) + \int_0^{N_S} L_S^{Rv}(u,t)du \tag{29}$$

$$L_B(t) = \int_0^{N_B} L_B^X(z,t)dz + L_B^{Rh}(t) + \int_0^{N_B} L_B^{Rv}(z,t)dz$$

After some calculation (see Appendix 6), we can express these both conditions as follows:

$$1 = \frac{\theta(\sigma-1)}{\sigma}\frac{D(t)}{w_S(t)L_S(t)} + \frac{\dot{N}_S(t)}{N_S(t)}\frac{N_S(t)}{\beta_S^h (K_S(t))^\gamma L_S(t)} + v_S(t) \tag{30}$$

$$1 = \frac{(1-\theta)(\sigma-1)}{\sigma}\frac{D(t)}{w_B(t)L_B(t)} + \frac{\dot{N}_B(t)}{N_B(t)}\frac{N_B(t)}{\beta_B^h (K_B(t))^\eta L_B(t)} + v_B(t)$$

Equilibrium Determination

We define the following stationary variables along the steady state:

$$c_B(t) = \frac{D(t)}{w_B(t)L_B(t)} \text{ and } c_S(t) = \frac{D(t)}{w_S(t)L_S(t)} \tag{31}$$

$$\kappa_B(t) = \frac{N_B(t)}{(K_B(t))^\eta L_B(t)} \text{ and } \kappa_S(t) = \frac{N_S(t)}{(K_S(t))^\eta L_S(t)} \tag{32}$$

$$v_B(t) = \frac{L_B^{Rv}(t)}{L_B(t)} \text{ and } v_S(t) = \frac{L_S^{Rv}(t)}{L_S(t)} \tag{33}$$

Dynamical System of the Model

Using these stationary variables we can expose the dynamical system of our model:

$$\frac{\dot{c}_B(t)}{c_B(t)} = \frac{(1-\theta)\beta_B^h c_B(t)}{\sigma\kappa_B(t)} - (\rho - n_B) + \frac{\dot{\kappa}_B(t)}{\kappa_B(t)} - \frac{\beta_B^h}{\kappa_B(t)}\left(1 - \frac{(1-\theta)(\sigma-1)}{\sigma}c_B(t) - v_B(t)\right) \tag{34}$$

$$\frac{\dot{c}_S(t)}{c_S(t)} = \frac{(1-\theta)\beta_S^h c_B(t)}{\sigma\kappa_S(t)} - (\rho - n_S) + \frac{\dot{\kappa}_S(t)}{\kappa_S(t)} - \frac{\beta_S^h}{\kappa_S(t)}\left(1 - \frac{\theta(\sigma-1)}{\sigma}c_S(t) - v_S(t)\right)$$

$$\frac{\dot{Q}_B(t)}{Q_B(t)} = (\lambda_B - 1)\beta_B^v \left(\frac{v_B(t)}{\kappa_B(t)}\right)^\delta \tag{35}$$

$$\frac{\dot{Q}_S(t)}{Q_S(t)} = (\lambda_S - 1)\beta_S^v \left(\frac{v_S(t)}{\kappa_S(t)}\right)^\mu$$

$$\frac{\dot{\kappa}_B(t)}{\kappa_B(t)} = \frac{\beta_B^h}{\kappa_B(t)}\left(1 - \frac{(1-\theta)(\sigma-1)}{\sigma}c_B(t) - v_B(t)\right) - \eta\frac{\dot{\kappa}_B(t)}{\kappa_B(t)} - n_B \tag{36}$$

$$\frac{\dot{\kappa}_S(t)}{\kappa_S(t)} = \frac{\beta_S^h}{\kappa_S(t)}\left(1 - \frac{\theta(\sigma-1)}{\sigma}c_S(t) - v_S(t)\right) - \gamma\frac{\dot{\kappa}_S(t)}{\kappa_S(t)} - n_S$$

We know that $\frac{\dot{D}(t)}{D(t)} = \frac{\dot{d}_B(t)}{d_B(t)} + n_B = \frac{\dot{d}_S(t)}{d_S(t)} + n_S$. It is required in order to have a steady state without divergence between both supplier and buyer, so that the stationary state variable growth rate C_B and is as follows:

$$\frac{\dot{c}_B(t)}{c_B(t)} = \frac{\dot{d}_B(t)}{d_B(t)} - \frac{\dot{w}_B(t)}{w_B(t)} \text{ and } \frac{\dot{c}_S(t)}{c_S(t)} = \frac{\dot{d}_S(t)}{d_S(t)} - \frac{\dot{w}_S(t)}{w_S(t)} \tag{37}$$

At the steady state equilibrium, the conditions $\frac{\dot{c}_B(t)}{c_B(t)} = 0$, a nd $\frac{\dot{c}_S(t)}{c_S(t)} = 0$ implies that consumption expenditures in buyer or supplier grow at the rate of growth of the wage rate.

Product Varieties Growth Rates

At the steady state, the growth rates of the number of varieties of products in supplier are (Appendix 3):

$$g_S^N = \frac{\gamma(1-\kappa_S)n_B + (1-\eta)(1-\kappa_B)n_S}{\eta\kappa_B(1-\gamma) + (1-\gamma\kappa_S)(1-\eta)} \tag{38}$$

with $= g_S^N = \frac{\dot{N}_S(t)}{N_S(t)}$, the growth rate of supplier's variable product varieties. Product varieties growth rates depend only on exogenous parameters and varieties growth requires population growth in at least supplier or buyer.

Symmetrically, at the steady state, the growth rates of the number of varieties of products in buyer are (Appendix 3):

$$Q_S(t) = N_S(t)^{-1} \int_0^{N_S} (q^S(u,t))^{\sigma-1} du \qquad (39)$$

$$g_S^Q = (\lambda_S - 1)\beta_S^v (\frac{v_S}{\kappa_S})^{\delta}$$

with = (β_S^v), the growth rate of buyer's variable product varieties. Product varieties growth rates depend only on exogenous parameters and varieties growth requires population growth in at least buyer or supplier.

Product Quality Growth Rates

The growth rates of the average weighted levels of quality in supplier *S* (Appendix 4):

$$Q_S(t) = N_S(t)^{-1} \int_0^{N_S} (q^S(u,t))^{\sigma-1} du \qquad (40)$$

$$g_S^Q = (\lambda_S - 1)\beta_S^v (\frac{v_S}{\kappa_S})^{\delta}$$

Product qualities growth rates depend on parameters that can be influenced by economic policy (β_S^v) so that a subsidy to vertical R& D can promote growth in the long run.

Symmetrically, the growth rates of the average weighted levels of quality in buyer *B* (Appendix 4):

$$Q_B(t) = N_B(t)^{-1} \int_0^{N_B} (q^B(z,t))^{\sigma-1} dz \qquad (41)$$

$$g_B^Q = (\lambda_B - 1)\beta_B^v (\frac{v_B}{\kappa_B})^{\delta}$$

Product qualities growth rates depend on parameters that can be influenced by economic policy (β_B^v) so that a subsidy to vertical R& D can promote growth in the long run.

Level of Share

Let's consider that the share $\theta(t)$ is not exogenous but depend on the relative average qualities level and number of varieties in each buyer and supplier.

$$\theta(t) = (\frac{N_S(t)Q_S(t)}{N_B(t)Q_B(t)}) \qquad (42)$$

It is straightforward to see that $\theta(t)$ will evaluate if the sum of one of the supplier's or buyer's product quality and product variety growth rates is superior to the one of the supplier's or buyer's.

$$\frac{\dot{\theta}(t)}{\theta(t)} = (g_S^N + g_S^Q) - (g_B^N + g_B^Q) \qquad (43)$$

Along the balanced growth equilibrium is constant and thus $\frac{\dot{\theta}(t)}{\theta(t)} = 0$ implies that

$$(g_S^N + g_S^Q) = (g_B^N + g_B^Q) \qquad (44)$$

The supplier or buyer with highest capacities in one of the two innovative sectors is thus forced to have the lowest capacities in the other innovative sector. This feature is important to understand the main results of this model in terms of the optimal research policy.

Product's Utility Growth Rate in Supplier or Buyer

The instantaneous flow of per capita product utility of the representative consumers of supplier or buyer (*S*, *B*) is (Appendix 5)

$$\log u_i = \Theta + \log d_i - \sigma\left((1-\theta)\log w_B + \theta \log w_S\right) + \frac{(1-\theta)}{\sigma-1}\log N_B$$
$$+ \frac{(1-\theta)}{\sigma-1}\log Q_B + \frac{\theta}{\sigma-1}\log N_S + \frac{\theta}{\sigma-1}\log Q_S \qquad (45)$$

With $\Theta = (1-\theta)\log(1-\theta) + \theta\log\theta - \sigma\log(\frac{\sigma}{\sigma-1})$

The growth rate of the instantaneous flow of per capita product utility in supplier or buyer (*S*, *B*) is given by:

$$g_i^u = g_i^d - \sigma\left((1-\theta)g_B^w + \theta g_S^w\right) + \frac{(1-\theta)\left(g_B^N + g_B^Q\right) + \theta(g_S^N + g_S^Q)}{\sigma-1} \qquad (46)$$

The inter temporal product utility is the discounted instantaneous product utility flows sum,

$U_i = \int_0^{\infty} e^{-(\rho-n_i)\tau}\log\left(u_i(\tau)\right)d\tau$, which can be again expressed as follows:

$$U_i = \frac{\log u_i}{\rho - n_i} + \frac{g_i^u}{(\rho-n_i)^2} \qquad (47)$$

At steady state, the inter temporal product utility function of the supplier or buyer *i* is as follows:

$$U_i(0) = \frac{\Theta + \log d_i - \sigma\left((1-\theta)\log w_B + \theta\log w_S\right)}{(\rho-n_i)}$$
$$+ \frac{g_i^d - \sigma\left((1-\theta)g_B^w + \theta g_S^w\right) + \frac{(1-\theta)\left(g_B^N + g_B^Q\right) + \theta(g_S^N + g_S^Q)}{\sigma-1}}{(\rho-n_i)^2}$$

Strategic Research Policy

In this section, we determinate the optimal research policy in supply chain collaboration. Suppler and buyer can select to subsidy either the vertical R& D sector, either the horizontal R& D sector or both sectors indifferently. The main policy is only subsidy to vertical R& D have long term effect on product development. We thus restrain our analysis to these subsidies at vertical research.

Table 3: Numerical Simulation Results

S.No.	Variables	Benchmark Simulation	Subsidy by Supplier $\tau_S^v = 0.15$	Subsidies by Buyer $\tau_S^v = 0.15$
1	Subsidy to keep θ constant	-	0.15	0.08
	Supplier's product varieties growth rate, g_S^N	3.0%	3.0%	3.0%
2	Buyer's product varieties growth rate, g_B^N	3.50%	3.50%	3.50%
3	Supplier, Average qualities growth rate, g_S^Q	1.10%	1.45%	1.25%
4	Buyer, Average qualities growth rate, g_B^Q	0.60%	0.80%	0.75%
6	Common product utility growth rate, g_i^u	1.60%	1.85%	1.75%
	Supplier, share of production labour, l_S^X	0.954	0.945	0.940
7	Buyer, share of production labour, l_B^X	0.95	0.90	0.90
8	Supplier, vertical innovation labour share	0.055	0.065	0.61
9	Buyer, vertical innovation labour share	0.044	0.055	0.045
10	Supplier, horizontal innovation labour share	0.225	0.155	0.155
11	Buyer, horizontal innovation labour share	0.095	0.085	0.075
12	Common product utility level, $u_i(0)$	3.850	4.200	4.350
13	Buyer's product utility level, $U_i(0)$	2.295	2.350	2.300
14	Supplier, consumption level, c_S	2.440	2.445	2.440
15	Buyer, consumption level, c_B	2.350	2.250	2.275
16	Ratio of relative, wage levels, $\frac{w_B}{w_S}$	1.25	1.25	1.25

Numerical Example Simulations

In our computer simulations, we use as benchmark parameter values, $\theta = 0.6$, $\rho = 0.06$, $\delta = 0.8$, $\mu = 0.5$, $n_s = 0.02$, $n_B = 0.02$, $L_S = 2$, $L_B(0) = 2$, $\beta_S^h = 0.66$, $\beta_S^v = 0.22$, $\beta_B^h = 0.88$, $\beta_B^v = 0.22$, $\varepsilon_S = 0.6$ ($k_S = 0.5$), $\varepsilon_B = 0.5$ ($k_B = 0.5$), $\gamma = 0.8$, $\eta = 0.80$, $\lambda_S = 2.2$, $\lambda_B = 2.0$.

Subsidies to Vertical R& D

We first discuss the effects of subsidy to vertical R& D in supplier and buyer. The results for a subsidy $\tau^v = 15\%$ are presented in Table 3.

The most important result put into the light by the models with two R& D sectors is that the vertical innovative sector is the engine of growth in supply chains, whereas the horizontal innovative sector of product development is bounded with decreasing returns. Hence the optimal policy to promote growth is to allow subsidy to vertical research. This result is still valid in supply chain collaboration framework. Moreover a subsidy to vertical R& D improves the shared level of utility. But it introduces a growth rate difference which can have strong effect on trade and wages. The growth rate of a supplier's or buyer's economy is the sum of its product quality and product variety growth rates. The growth rates (product quality and product variety growth rates) have to be equals between the both supplier and buyer in order to ensure a balanced growth path with a constant fraction of buyer's expenses devoted products of each supplier and buyer. A policy that promotes growth has also effect on growth and relative wage. The supplier or buyer that initially has not conducted an innovation policy is forced to subsidy its research to development in order not to conserve it's competitively. We proceed to some numerical simulations of our theoretical model in order to shed light on some interesting results. Our main results are the followings:

(i) Starting from an balanced growth equilibrium with differentiated innovative capacities in supply chain collaboration between the supplier and buyer, a subsidy to vertical R& D have different effect depending on the vertical innovation capacities of the supplier or buyer that first introduces this policy. A subsidy to vertical R& D increases more the common growth rate when it is introduced by the supplier or buyer with the stronger vertical innovation capacities.

(ii) In terms of buyer's utility, subsidies to vertical R& D have positive effect (it increases the level of the inter-temporal utility). A subsidy to vertical R& D improves more utility when it is introduced by the supplier or buyer with the stronger vertical innovation capacities.

(iii) The supplier or buyer, which has not introduced a subsidy to vertical R& D is forced to do it in order to conserve the initial market shares in differentiated products. The level of the subsidy enabling to conserve the initial level of market shares will be higher for supplier or buyer, which is the one with the highest capacities in vertical R& D.

Another Innovation Policy

Finally, we can also investigate the effect of another policy enabling to promote qualitative growth: a policy that increases the average size of innovation (for example this policy can be implemented by improving the patent novelty requirement). This policy ameliorates the quality growth rate by increasing the size rather than the frequency of innovation.

Discussion

This paper makes contributions to the understanding of a supply chain collaboration dynamic model with two innovative R& D sectors in each supplier and buyer: a vertical R& D sector that improves the quality of existing differentiated products and a horizontal R& D sector that creates new differentiated products. The contribution involves the integrative approach to the supplier and buyer exchange differentiated products and beneficiate from knowledge spillovers (possibly impulse by R& D subsidies). While different streams of collaboration for product development have been previously studied as is shown by a great variety of literature, involving product variety, product structure and innovativeness, and design trade-off methodology, these literatures do not have attempt on R& D policy. Most of previous research simply blurs or subordinates other instruments. In previous literature, there is a partial attempt to bring them together and thereby they provide overlapping focused instruments in the attempt to improve supply chain performance. This paper has contribution for the long term policy effects of R& D subsidies in this context had been studied in this paper. In this contribution, we have realized an attempt to integrate the product development model in a supply chain collaboration framework. This paper enables us to discuss of the optimal research policy integrating some feedback effects from innovation and knowledge spillovers. Our main result is that the effect of a subsidy to vertical R& D (the only subsidy that has a long term effect) depends on the relative innovative capacities of the supplier or buyer that realized this policy. The model proposed in this research provides a more comprehensive approach than does much of the work on review studies of collaboration in integrating diverse literature to identify and unify different instruments in supply chain collaboration.

This paper provides different contributions for model development for supply chain collaboration in equilibrium conditions. This paper contributes by specifying buyer's product utility function, supplier's and buyer's demand function of differentiated products, and supplier and buyer's profit functions. The system equations for both product innovations sectors of supplier and buyer will be developed in this paper. This paper also developed the model for innovation values of supplier and buyer. This paper also examines the equilibrium determination, dynamical system of the model; varieties growth rates of products, level for relative average qualities and number of varieties and buyer's product utility growth rate. This paper determines the optimal research policy in supply chain collaboration. A numerical example simulation will illustrate subsidy either the vertical R& D sector, either the horizontal R& D sector or both sectors indifferently and results are discussed.

This research provides several implications for future research. The mechanisms for supply chain collaboration for product development need to be studied in detail. The collaboration mechanisms can further be of different sub types. To collaborate the whole supplychain, the aggregation of the impact of all collaboration mechanisms on the performance of supply chain is required. Various combinations may be explored with the help of simulation. Supply chain contracts have proved to coordinate single period supply chains. The research is required to explore the utility of contracts in multi-period cases. In multi period model, the supply chain members are more expose to the uncertainty as they are dealing with supply chain members frequently. How various collaboration mechanisms can be allied in multi period problems as well as can we evaluate collaboration in such case?

Very few studies have been reported to quantify risk or uncertainty in supply chain collaboration for product development. The Bullwhip effect has extensively been discussed in the literature. Actually, there can be many variations seen in supply chain like supply uncertainty, delay in delivery having cascading effect as we go downwards in the supply chain, which is similar to the order variation in Bullwhip effect. How supply chain collaboration can help in mitigating such uncertainties for product development is one of the important research issues?

The model proposed put forth in this work create a path for future studies. Managers could use this insight. To make trade-off decisions regarding whether investing in developing advanced stages of collaboration would be justified by the potential benefits and accordingly, could use it as a guide for prioritizing investments in collaboration. Academics could use the information to further advance the study of collaborative structures between trading partners. Managers may employ a mix of collaboration types across a range of products by determining which type of collaboration best fits their business model. Managers need to consider that while collaborative transaction management may be the best fit for standard items, collaborative event management may be the best option for promotional items and collaborative process management may be required for highly innovative items with high demand uncertainty.

Conclusion

It is believed that through the supply chain collaboration in product development model, a basic conceptual structure has been generated that can help the connection between the supply chain management process and the product development management process. The objective of this paper is within the theoretical realm: the possibility of there being a theoretical reference that supplies information on how to involve supply chain collaboration in product development, in other words, building this reference is the main point of the ongoing study. Buyer and supplier involvement is an interesting issue. Involvement of a buyer brings time efficiencies such that it reduces product development time. The “customer is king” attitude has been a driving force for many businesses since the early 1980s. With the introduction of supply-chain management practices, the direction of this relationship has started to change such that more emphasis has been given to the relationship with suppliers and, ultimately, a business-to-business (b2b) concept has emerged. The innovation based growth literature seems to have converged towards a unified supply chain collaboration framework with two innovative sectors. The long term policy effects of tax or subsidy in this context had not been studied. In this contribution, we have realised an attempt to integrate the product development model in a supply chain collaboration framework. This enables us to discuss of the optimal research policy integrating some feedback effects from innovation and knowledge spillovers. Our main result is that the effect of a subsidy to vertical R& D (the only subsidy that has a long term effect) depends on the relative innovative capacities of the supplier or buyer that realised this policy. If the supplier or buyer is not the one with the best capacities in the vertical innovation sector, then this policy has negative effect both on product development if the buyer or supplier does not equally introduce a subsidy to R& D. This result is important since it forces authorities to appreciate their innovative capacities before introducing R & Dsubsidies (the positive effect of this policy is not ensured as in the supply chain collaboration framework). If the supplier or buyer is not the one with the best capacities in the vertical innovative sector, it has to implement another policy before subsidizing its vertical innovative sector if it wants this latest policy to be efficient.

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APPENDIX 1: DEMAND FUNCTION

$x_B(z,t)$and$x_S(u,t)$:

$$\max u_i(t) = \left(\int_0^{N_B} \left(x_B(z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}} dz\right)^{\frac{\sigma(1-\theta)}{\sigma-1}} \left(\int_0^{N_S} \left(x_S(u,t)q^S(u,t)\right)^{\frac{\sigma-1}{\sigma}} du\right)^{\frac{\sigma\theta}{\sigma-1}}$$

$$\text{u.c. } d_i == \int_0^{N_B} p^B(z,t)x_B(z,t)\,dz + \int_0^{N_S} p^S(u,t)x_S(u,t)\,du$$

$$(1-\theta)\frac{\left(\int_0^{N_B}\left(x_B(z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}dz\right)^{\frac{\sigma(1-\theta)}{\sigma-1}-1}}{\left(\int_0^{N_S}\left(x_S(u,t)q^S(u,t)\right)^{\frac{\sigma-1}{\sigma}}du\right)^{-\frac{\sigma\theta}{\sigma-1}}}\frac{\left(x_B(z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}}{x_B(z,t)} = \lambda p_B(z,t)$$

$$\theta\frac{\left(x_S(u,t)q^S(u,t)\right)^{\frac{\sigma-1}{\sigma}}}{x_S(u,t)}\frac{\left(\int_0^{N_S}\left(x_S(u,t)q^S(u,t)\right)^{\frac{\sigma-1}{\sigma}}du\right)^{\frac{\sigma\theta}{\sigma-1}-1}}{\left(\int_0^{N_B}\left(x_B(Z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}dz\right)^{-\frac{\sigma(1-\theta)}{\sigma-1}}} = \lambda p_S(u,t)$$

$$\left(x_B(z,t)\right)^{-\frac{1}{\sigma}} = \frac{\lambda p_B(z,t)\left(q^B(z,t)\right)^{-\frac{\sigma-1}{\sigma}}}{(1-\theta)}\frac{\left(\int_0^{N_B}\left(x_B(Z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}dz\right)^{1-\frac{\sigma(1-\theta)}{\sigma-1}}}{\left(\int_0^{N_S}\left(x_S(u,t)q^S(u,t)\right)^{\frac{\sigma-1}{\sigma}}du\right)^{-\frac{\sigma\theta}{\sigma-1}}}$$

$$\left(x_S(u,t)\right)^{-\frac{1}{\sigma}} = \frac{\lambda p_S(u,t)\left(q^S(u,t)\right)^{-\frac{\sigma-1}{\sigma}}}{\theta}\frac{\left(\int_0^{N_S}\left(x_S(u,t)q^S(u,t)\right)^{\frac{\sigma-1}{\sigma}}du\right)^{1-\frac{\sigma\theta}{\sigma-1}}}{\left(\int_0^{N_B}\left(x_B(z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}dz\right)^{\frac{\sigma(1-\theta)}{\sigma-1}}}$$

$$x_B(z,t) = \frac{\lambda^{-\sigma}p_B(z,t)^{-\sigma}\left(q^B(z,t)\right)^{\sigma-1}}{(1-\theta)^{-\sigma}}\frac{\left(\int_0^{N_B}\left(x_B(z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}dz\right)^{\frac{\sigma^2(1-\theta)}{\sigma-1}-\sigma}}{\left(\int_0^{N_S}\left(x_S(u,t)q^S(u,t)\right)^{\frac{\sigma-1}{\sigma}}du\right)^{-\frac{\sigma^2\theta}{\sigma-1}}}$$

$$x_S(u,t) = \frac{\lambda^{-\sigma}p_S(u,t)^{-\sigma}\left(q^S(u,t)\right)^{\sigma-1}}{\theta^{-\sigma}}\frac{\left(\int_0^{N_S}\left(x_S(u,t)q^S(u,t)\right)^{\frac{\sigma-1}{\sigma}}du\right)^{\frac{\sigma^2\theta}{\sigma-1}-\sigma}}{\left(\int_0^{N_B}\left(x_B(z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}dz\right)^{-\frac{\sigma^2(1-\theta)}{\sigma-1}}}$$

$$\frac{\int_0^{N_B} p_B(z,t)x_B(z,t)dz}{\left(\int_0^{N_B}\left(p_B(z,t)\right)^{1-\sigma}\left(q^B(z,t)\right)^{\sigma-1}dz\right)} = \frac{\lambda^{-\sigma}}{(1-\theta)^{-\sigma}}\frac{\left(\int_0^{N_S}\left(x_S(u,t)q^S(u,t)\right)^{\frac{\sigma-1}{\sigma}}du\right)^{\frac{\sigma^2\theta}{\sigma-1}}}{\left(\int_0^{N_B}\left(x_B(z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}dz\right)^{\sigma-\frac{\sigma^2(1-\theta)}{\sigma-1}}}$$

$$\frac{\int_0^{N_S} p_S(u,t)x_S(u,t)du}{\left(\int_0^{N_S}\left(p_S(u,t)\right)^{1-\sigma}\left(q^S(u,t)\right)^{\sigma-1}du\right)} = \frac{\lambda^{-\sigma}}{\theta^{-\sigma}}\frac{\left(\int_0^{N_B}\left(x_B(z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}dz\right)^{\frac{\sigma^2(1-\theta)}{\sigma-1}}}{\left(\int_0^{N_S}\left(x_S(u,t)q^S(u,t)\right)^{\frac{\sigma-1}{\sigma}}du\right)^{\sigma-\frac{\sigma^2\theta}{\sigma-1}}}$$

and thus $x_B(z,t)$ and $x_S(u,t)$ are:

$$x_B(z,t) = \frac{\left(p_B(z,t)\right)^{-\sigma}\left(q_B(z,t)\right)^{\sigma-1}\left(\int_0^{N_B} p_B(z,t)q_B(z,t)dz\right)}{\left(\int_0^{N_B}\left(p_B(z,t)\right)^{-\sigma}\left(q_B(z,t)\right)^{\sigma-1}dz\right)}$$

$$x_S(u,t) = \frac{\left(p_S(u,t)\right)^{-\sigma}\left(q_S(u,t)\right)^{\sigma-1}\left(\int_0^{N_S} p_S(u,t)q_S(u,t)du\right)}{\left(\int_0^{N_S}\left(p_S(u,t)\right)^{-\sigma}\left(q_S(u,t)\right)^{\sigma-1}du\right)}$$

Appendix 2

We can easily verify that $\theta = \frac{\int_0^{N_B} p^B(z,t)x_B(z,t)dz}{\int_0^{N_B} p^B(z,t)x_B(z,t)dz + \int_0^{N_S} p^S(u,t)x_S(u,t)du}$

We first use the following expression given by the first order conditions: =

$$\left(\int_0^{N_B} p^B(z,t)\,x_B(z,t)dz\right)^{\frac{1}{\sigma}} = \frac{(1-\theta)\left(\int_0^{N_B} p^B(z,t)^{1-\sigma}\,(q_B(z,t))^{\sigma-1}dz\right)^{\frac{1}{\sigma}}}{\lambda}\frac{\left(\int_0^{N_S} x_S(u,t)q^S(u,t)^{\frac{\sigma-1}{\sigma}}du\right)^{\frac{\sigma\theta}{\sigma-1}}}{\left(\int_0^{N_B}\left(x_B(z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}dz\right)^{1-\frac{\sigma(1-\theta)}{\sigma-1}}}$$

$$\left(\int_0^{N_S} p^S(u,t)\,x_S(u,t)du\right)^{\frac{1}{\sigma}} =$$

$$\frac{\theta\left(\int_0^{N_S} p^S(u,t)^{1-\sigma}\,(q_S(u,t))^{\sigma-1}du\right)^{\frac{1}{\sigma}}}{\lambda}\frac{\left(\int_0^{N_B}\left(x_B(z,t)q^B(z,t)\right)^{\frac{\sigma-1}{\sigma}}dz\right)^{\frac{\sigma(1-\theta)}{\sigma-1}}\left(\int_0^{N_S} x_S(u,t)q^S(u,t)^{\frac{\sigma-1}{\sigma}}du\right)^{\frac{\sigma\theta}{\sigma-1}}}{\left(\int_0^{N_S} x_S(u,t)q^S(u,t)^{\frac{\sigma-1}{\sigma}}du\right)^{1-\frac{\sigma\theta}{\sigma-1}}}$$

And then substitute $x_B(z, t)$ and $x_S(u, t)$ by their expression, so that we get:

$$\frac{\left(\int_0^{N_B} p^B(z,t)x_B(z,t)dz\right)^{\theta}}{\left(\int_0^{N_S} p^S(u,t)x_S(u,t)du\right)^{-\theta}} = \frac{(1-\theta)}{\lambda}\frac{\left(\int_0^{N_S} p^S(u,t)^{1-\sigma}\,(q_S(u,t))^{\sigma-1}du\right)^{\frac{\sigma}{\sigma-1}}}{\left(\int_0^{N_B}\left(p^B(z,t)^{1-\sigma}\,q^B(z,t)\right)^{\sigma-1}dz\right)^{-\frac{(1-\theta)}{\sigma-1}}}$$

$$\frac{\left(\int_0^{N_S} p^S(u,t)x_S(u,t)du\right)^{1-\theta}}{\left(\int_0^{N_B} p^B(z,t)x_B(z,t)dz\right)^{\theta-1}} = \frac{\theta}{\lambda}\frac{\left(\int_0^{N_S} p^S(u,t)^{1-\sigma}\,(q_S(u,t))^{\sigma-1}du\right)^{\frac{\sigma}{\sigma-1}}}{\left(\int_0^{N_B}\left(p^B(z,t)^{1-\sigma}\,q^B(z,t)\right)^{\sigma-1}dz\right)^{-\frac{(1-\theta)}{\sigma-1}}}$$

Making the ratio of these two expression give

$$\frac{\int_0^{N_B} p^B(z,t)x_B(z,t)dz}{\int_0^{N_S} p^S(u,t)x_S(u,t)du} = \frac{(1-\theta)}{\theta}$$

So that θ is effectively:

$$\theta = \frac{\int_0^{N_S} p^S(u,t)x_S(u,t)du}{\int_0^{N_S} p^B(z,t)x_B(z,t)dz + \int_0^{N_B} p^S(u,t)x_S(u,t)du}$$

Appendix 3: Product varieties growth rates and

$$\frac{\dot{N}_B(t)}{N_B(t)} = \beta_B^h (N_B(t))^{\eta(1-\kappa_B)-1}(N_S(t))^{\eta\kappa_B} L_B^{Rh}(t)$$

$$\frac{d\log\left(\frac{\dot{N}_B(t)}{N_B(t)}\right)}{dt} = 0, \text{ Hence } \left(1-\eta(1-\kappa_B)\right)g_B^N - \eta\kappa_B g_S^N = n_B$$

$$\frac{\dot{N}_S(t)}{N_S(t)} = \beta_S^h (N_S(t))^{\eta(1-\kappa_S)}(N_B(t))^{\eta\kappa_S-1} L_S^{Rh}(t)$$

$$\frac{d\log\left(\frac{\dot{N}_S(t)}{N_S(t)}\right)}{dt} = 0, \text{ Hence } \left(1-\gamma(1-\kappa_S)\right)g_S^N - \gamma(1-\kappa_S)g_B^N = n_S$$

$$\left(1-\eta(1-\kappa_B)\right)g_B^N - \eta\kappa_B g_S^N = n_B$$

$$\left(1-\gamma(1-\kappa_S)\right)g_S^N - \gamma(1-\kappa_S)g_B^N = n_S$$

$$g_B^N = \frac{n_B}{(1-\eta(1-\kappa_B))} + \frac{\eta\kappa_B}{(1-\eta(1-\kappa_B))}g_S^N$$

$$g_S^N = \frac{(1-\gamma\kappa_S)}{\gamma(1-\kappa_S)}g_S^N - \frac{n_S}{\gamma(1-\kappa_S)}$$

$$g_S^N = \frac{\frac{n_B\gamma(1-\kappa_S) + \left(1-\eta(1-\kappa_B)\right)n_S}{\left(1-\eta(1-\kappa_B)\right)\gamma(1-\kappa_S)}}{\left(\frac{(1-\gamma\kappa_S)\left(1-\eta(1-\kappa_B)\right) - \eta\kappa_{B\gamma}(1-\kappa_S)}{\gamma(1-\kappa_S)\left(1-\eta(1-\kappa_B)\right)}\right)}$$

$$g_S^N = \frac{\gamma(1-\kappa_S)n_B + (1-\eta)(1-\kappa_B)n_S}{\eta\kappa_B(1-\gamma) + (1-\gamma\kappa_S)(1-\eta)}$$

$$g_B^N = \frac{n_B(1-\gamma\kappa_S) + \eta\kappa_B n_S}{\eta\kappa_B(1-\gamma) + (1-\gamma\kappa_S)(1-\eta)}$$

Appendix 4: Product quality growth rates

Taking $\left(q^i(z,t)\right)^{\sigma-1} = (\lambda_i)^{m(z,t)}$, where $m(z, t)$ is the number of innovations that has occurred at time t in sector z, the average level of weighted qualities in supplier or buyer (S, B) is:

$$Q_i(t) = N_i(t)^{-1}\int_0^{N_i}(q^i(z,t))^{\sigma-1}dz = N_i(t)^{-1}\int_0^{N_i}(\lambda_i)^{m(z,t)}dz$$

The variation of the average level of weighted qualities at time is thus:

$$\dot{Q}_i(t) = N_i(t)^{-1}\int_0^{N_i}((\lambda_i)^{m(z,t)+1} - (\lambda_i)^{m(z,t)})\,\phi_i(z,t)dz = (\lambda_i - 1)\phi_B(t)Q_i(t)$$

Given that $\phi_B(z,t) = \phi_B(t)$ and $\phi_B(t) = \beta_B^v\left(K_B(t)\right)^{\eta\delta}\left(\frac{L_B^{Rv}(z,t)}{N_B(t)}\right)^{\delta}$, we can express the growth rate of the average weighted quality as follows:

$$\frac{\dot{Q}_B(t)}{Q_B(t)} = (\lambda_B - 1)\beta_B^v\left(\frac{\left(K_B(t)\right)^{\eta}L_B^{Rv}(t)}{N_B(t)}\right)^{\delta}$$

Taking the following stationary variables along the balanced growth path $\kappa_S = \frac{N_S(t)}{\left(K_S(t)\right)^{\eta}L_S(t)}$ and $v_S = \frac{L_S^{Rv}}{L_S}$, the growth rate of quality of supplier becomes:

$$g_S^Q = \frac{\dot{Q}_S(t)}{Q_S(t)} = (\lambda_S - 1)\beta_S^v\left(\frac{v_S(t)}{\kappa_S(t)}\right)^{\mu}$$

Symmetrically, taking the following variables that will be stationary along the balanced growth path $\frac{N_B(t)}{\left(K_B(t)\right)^{\eta}L_B(t)}$ and $v_B = \frac{L_B^{Rv}}{L_B}$, the growth rate of quality of buyer becomes:

$$g_B^Q = \frac{\dot{Q}_B(t)}{Q_B(t)} = (\lambda_B - 1)\beta_B^v\left(\frac{v_B(t)}{\kappa_B(t)}\right)^{\delta}$$

Appendix 5: Determination of the product utility function of supplier and buyer

Along the balanced growth path, the flow of utility is given by:

$$u_i = \left(\int_0^{N_B}\left(\frac{(p^B(z))^{-\sigma}\left(q^B(z)\right)^{\sigma}(1-\theta_i)d_i}{\left(\int_0^{N_B}\left(p^B(z)\right)^{1-\sigma}\left(q^B(z)\right)^{\sigma-1}dz\right)}\right)^{\frac{1-\sigma}{\sigma}}dz\right)^{\frac{\sigma(1-\theta_S)}{\sigma-1}}$$

$$\left(\int_0^{N_S}\left(\frac{(p^S(u))^{-\sigma}\left(q^S(u)\right)^{\sigma}\theta_i d_i}{\left(\int_0^{N_s}\left(p^s(u)\right)^{1-\sigma}\left(q^s(u)\right)^{\sigma-1}du\right)}\right)^{\frac{\sigma-1}{\sigma}}du\right)^{\frac{\sigma\theta}{\sigma-1}}$$

$$u_i=\left(\frac{\sigma}{\sigma-1}w_B\right)^{-\sigma(1-\theta)}\left(\int_0^{N_B}(q^B(z))^{\sigma-1}dz\right)^{\frac{(1-\theta)}{\sigma-1}}((1-\theta)d_i)^{(1-\theta)}$$

$$\left(\frac{\sigma}{\sigma-1}w_S\right)^{-\sigma\theta}\left(\int_0^{N_S}(q^S(u))^{\sigma-1}du\right)^{\frac{\theta}{\sigma-1}}(\theta d_i)^{\theta}$$

Given $Q_i(t)= N_i(t)^{-1}\int_0^{N_i}(q^i(z,t))^{\sigma-1}dz$, this last expression can be expressed as follows:

$$u_i=((1-\theta)d_i)^{(1-\theta)}\left(\frac{\sigma}{\sigma-1}w_B\right)^{-\sigma(1-\theta)}(N_BQ_B)^{\frac{(1-\theta)}{\sigma-1}}\left(\frac{\sigma}{\sigma-1}w_S\right)^{-\sigma\theta}(N_SQ_S)^{\frac{\theta}{\sigma-1}}(\theta d_i)^{\theta}$$

$$\log u_i=(1-\theta)\log(1-\theta)+\theta\log\theta-\sigma\log\left(\frac{\sigma}{\sigma-1}\right)$$

$$+\log d_i-\sigma\left((1-\theta)\log w_B+\theta\log w_S\right)+\frac{(1-\theta)}{\sigma-1}\log N_B$$

$$+\frac{(1-\theta)}{\sigma-1}\log Q_B+\frac{\theta}{\sigma-1}\log N_S+\frac{\theta}{\sigma-1}\log Q_S$$

So that the growth rate of the instantaneous flow of per capita product utility in supplier orbuyer$i$ (S,B) is given by:

$$g_i^u=g_i^d-\sigma\left((1-\theta)g_B^w+\theta g_S^w\right)+\frac{(1-\theta)}{\sigma-1}\left(g_B^N+g_B^Q\right)+\frac{\theta}{\sigma-1}\left(g_S^N+g_S^Q\right)$$

Appendix 6: Labour used in manufacturing of supplier and buyer (,)

$$L_B^X(t)=\int_0^{N_B}L_B^X(z,t)dz$$
$$=\int_0^{N_B}(x_B^H(z,t)+x_S^H(z,t))dz$$

$$=(1-\theta)D(t)\int_0^{N_B}\frac{\left(\frac{\sigma}{\sigma-1}\right)^{-\sigma}(w_B(t))^{-\sigma}(q^B(z,t))^{\sigma-1}}{\left(\frac{\sigma}{\sigma-1}\right)^{1-\sigma}(w_B(t))^{1-\sigma}\int_0^{N_B}(q^B(z,t))^{\sigma-1}dz}dz$$

$$=\frac{(1-\theta)D(t)}{\left(\frac{\sigma}{\sigma-1}\right)w_B(t)}$$

Appendix 7: Product innovation values for supplier and buyer

The horizontal innovation values for supplier and buyer are as follows:

$$V_S^h(u,t) = \frac{w_S(t)(1-r_S^h)}{\beta_S^h(K_S(t))^{\gamma}}$$

$$V_B^h(z,t) = \frac{w_B(t)(1-r_B^h)}{\beta_B^h(K_B(t))^{\eta}}$$

So that the horizontal innovation values growth rate are:

$$\frac{\dot{V}_S^h(t)}{V_S^h(t)} = \frac{\dot{w}_S(t)}{w_S(t)} - \gamma\frac{\dot{K}_S(t)}{K_S(t)}$$

$$\frac{\dot{V}_B^h(t)}{V_B^h(t)} = \frac{\dot{w}_B(t)}{w_B(t)} - \eta\frac{\dot{K}_B(t)}{K_B(t)}$$

The vertical innovation values for supplier and buyer are as follows:

$$V_S^v(u,t) = \frac{w_S(t)(1-r_S^v)\left(q^S(u,t)\right)^{\sigma-1}(L_S^{Rv}(t))^{1-\mu}}{\beta_S^v Q_S(\mathrm{t})(N_S(\mathrm{t}))^{1-\mu}(K_S(t))^{\gamma\mu}}$$

$$V_B^v(z,t) = \frac{w_B(t)(1-r_B^v)\left(q^B(z,t)\right)^{\sigma-1}(L_B^{Rv}(t))^{1-\delta}}{\beta_B^v Q_B(\mathrm{t})(N_B(\mathrm{t}))^{1-\delta}(K_B(t))^{\eta\delta}}$$

So that the vertical innovation values growth rate are:

$$\frac{\dot{V}_S^v(u,t)}{V_S^v(u,t)} = \frac{\dot{w}_S(t)}{w_S(t)} + (\sigma-1)E\left[\frac{\dot{q}^S(u,t)}{q^S(u,t)}\right] + (1-\mu)n_S - \frac{\dot{Q}_S(t)}{Q_S(t)} - (2-\mu)\frac{\dot{N}_S(t)}{N_S(t)} - \gamma\mu\frac{\dot{K}_S(t)}{K_S(t)}$$

Since is constant during an innovation race if there is no innovation but can increase by a factor if an innovation occurs by supplier. Symmetrically, the vertical innovation values growth rate of buyer are

$$\frac{\dot{V}_B^v(z,t)}{V_B^v(z,t)} = \frac{\dot{w}_B(t)}{w_B(t)} + (\sigma-1)E\left[\frac{\dot{q}^B(z,t)}{q^B(z,t)}\right] + (1-\delta)n_B - \frac{\dot{Q}_B(t)}{Q_B(t)} - (1-\delta)\frac{\dot{N}_B(t)}{N_B(t)} - \eta\delta\frac{\dot{K}_B(t)}{K_B(t)}$$

$$E\left(\frac{\dot{V}_S^v(u,t)}{V_S^v(u,t)}\right) = \frac{\dot{w}_S(t)}{w_S(t)} + (1-\mu)n_S - \frac{\dot{Q}_S(t)}{Q_S(t)} - (2-\mu)\frac{\dot{N}_S(t)}{N_S(t)} - \gamma\mu\frac{\dot{K}_S(t)}{K_S(t)}$$

$$E\left(\frac{\dot{V}_B^v(z,t)}{V_B^v(z,t)}\right) = \frac{\dot{w}_B(t)}{w_B(t)} + (1-\delta)n_B - (1-\delta)\frac{\dot{N}_B(t)}{N_B(t)} - \eta\delta\frac{\dot{K}_B(t)}{K_B(t)}$$

with$E\left[\frac{\dot{q}^B(z,t)}{q^B(z,t)}\right] = \frac{1}{(\sigma-1)}\frac{\dot{Q}_B(z,t)}{Q_B(z,t)}$ and $E\left[\frac{\dot{q}^S(u,t)}{q^S(u,t)}\right] = \frac{1}{(\sigma-1)}\frac{\dot{Q}_S(t)}{Q_S(t)}$

Appendix 8: Dividend rates

Using expressions of profits flows and innovations values we can expressed the four dividend rates:

$$\frac{\pi_S^h(t)}{V_S^h(t)} = \frac{(1-\theta)}{(1-\tau_S^h)} \frac{\beta_S^h K_S(t)^\eta}{\sigma N_S(t) w_S(t)} D(t) = \frac{(1-\theta)}{(1-\tau_S^h)} \frac{\beta_S^h}{\sigma K_S(t)} \frac{D(t)}{w_S(t) L_S(t)}$$

$$\frac{\pi_S^h(t)}{V_S^h(t)} = \frac{(1-\theta)}{(1-\tau_S^v)} \frac{\beta_S^h K_S(t)^{\gamma\mu} L_S(t)^\mu v_S(t)^{\mu-1}}{\sigma N_S(t)^\mu} \frac{D(t)}{w_S(t) L_S(t)} = \frac{(1-\theta)}{(1-\tau_S^v)} \frac{\beta_S^h v_S(t)^{\mu-1}}{\sigma k_S(t)^\mu} \frac{D(t)}{w_S(t) L_S(t)}$$

$$\frac{\pi_B^h(t)}{V_B^h(t)} = \frac{(1-\theta)}{(1-\tau_B^h)} \frac{\beta_B^h K_B(t)^\eta}{\sigma N_B(t) w_B(t)} D(t) = \frac{(1-\theta)}{(1-\tau_B^h)} \frac{\beta_B^h}{\sigma K_B(t)} \frac{D(t)}{w_B(t) L_B(t)}$$

$$\frac{\pi_B^h(t)}{V_B^h(t)} = \frac{(1-\theta)}{(1-\tau_B^v)} \frac{\beta_B^h K_B(t)^{\eta\delta} L_B(t)^\delta v_B(t)^{\delta-1}}{\sigma N_B(t)^\delta} \frac{D(t)}{w_B(t) L_B(t)} = \frac{(1-\theta)}{(1-\tau_B^v)} \frac{\beta_B^h v_B(t)^{\delta-1}}{\sigma k_B(t)^\delta} \frac{D(t)}{w_B(t) L_B(t)}$$

Appendix 9: Non arbitrage-conditions

The no-arbitrage condition in the horizontal research sector of supplier S is:

$$r(t) + \phi_S(t) = \frac{(1-\theta)}{(1-\tau_S^h)} \frac{\beta_S^h c_S(t)}{\sigma \kappa_S(t)} + \frac{\dot{w}_S(t)}{w_S(t)} - \gamma \frac{\dot{K}_S(t)}{K_S(t)}$$

The no-arbitrage condition in the horizontal research sector of buyer B is:

$$r(t) + \phi_B(t) = \frac{(1-\theta)}{(1-\tau_B^h)} \frac{\beta_B^h c_B(t)}{\sigma \kappa_B(t)} + \frac{\dot{w}_B(t)}{w_B(t)} - \eta \frac{\dot{K}_B(t)}{K_B(t)}$$

The no-arbitrage condition in the vertical research sector in supplier S is:

$$r(t) + \phi_S(t) = \frac{\theta}{(1-\tau_S^v)} \frac{\beta_S^v v_S(t)^{\mu-1}}{\sigma \kappa_S(t)^\mu} c_S(t) + \frac{\dot{w}_S(t)}{w_S(t)} + (1-\mu) n_S - (1-\mu) \frac{\dot{N}_S(t)}{N_S(t)} \gamma\mu \frac{\dot{K}_S(t)}{K_S(t)}$$

The no-arbitrage condition in the vertical research sector of buyer B is:

$$r(t) + \phi_B(t) = \frac{(1-\theta)}{(1-\tau_B^v)} \frac{\beta_B^h v_B(t)^{\delta-1}}{\sigma \kappa_B(t)^\delta} c_B(t) + \frac{\dot{w}_B(t)}{w_B(t)} + (1-\delta) n_B - (1-\delta) \frac{\dot{N}_B(t)}{N_B(t)} - \eta\delta \frac{\dot{K}_B(t)}{K_B(t)}$$

Appendix 10: Steady state values of variables , and

We establish the steady state equilibrium values of the stationary variables c_i, k_i and v_i. First, by equalizing the two no-arbitrage conditions we obtain the equalities below:

$$\frac{\beta_S^v}{(1-\tau_S^v)} \left(\frac{v_S(t)}{\kappa_S(t)}\right)^\mu - \frac{\beta_S^h}{(1-\tau_S^h)} \left(\frac{v_S(t)}{k_S(t)}\right) = \frac{v_S(t)\sigma(1-\mu)}{c_S(t)\theta} \frac{\dot{k}_S(t)}{k_S(t)}$$

$$\frac{\beta_B^v}{(1-\tau_B^v)} \left(\frac{v_B(t)}{\kappa_B(t)}\right)^\delta - \frac{\beta_B^h}{(1-\tau_B^h)} \left(\frac{v_B(t)}{k_B(t)}\right) = \frac{v_B(t)\sigma(1-\sigma)}{c_B(t)(1-\theta)} \frac{\dot{k}_B(t)}{k_B(t)}$$

Omitting time and remembering that $\frac{\dot{k}_i}{k_i} = 0$, we have the following relation between u and k at the steady state:

$$v_S^* = k_S^* \left(\frac{\beta_S^v}{\beta_S^h}\frac{(1-\tau_S^h)}{(1-\tau_S^v)}\right)^{\frac{1}{1-\mu}}$$

$$v_B^* = k_B^* \left(\frac{\beta_B^v}{\beta_B^h}\frac{(1-\tau_B^h)}{(1-\tau_B^v)}\right)^{\frac{1}{1-\delta}}$$

Second, using the horizontal no-arbitrage conditions, the full –employment conditions, we can express the following dynamic equations:

$$\frac{\dot{c}_S(t)}{c_S(t)} = \frac{\theta \beta_S^h c_S(t)}{\sigma(1-\tau_S^h)\kappa_S(t)} - (\rho - n_S) + \frac{\dot{k}_S(t)}{k_S(t)} - \frac{\beta_S^h}{\kappa_S(t)}\left(1 - \frac{(1-\theta)(\sigma-1)}{\sigma} c_S(t) - v_S(t)\right)$$

$$\frac{\dot{c}_B(t)}{c_B(t)} = \frac{(1-\theta)\beta_B^h c_B(t)}{\sigma(1-\tau_B^h)\kappa_B(t)} - (\rho - n_B) + \frac{\dot{k}_B(t)}{k_B(t)} - \frac{\beta_B^h}{\kappa_B(t)}\left(1 - \frac{(1-\theta)(\sigma-1)}{\sigma} c_B(t) - v_B(t)\right)$$

At the steady state $\frac{\dot{c}_B}{C_B} = \frac{\dot{c}_S}{C_S} = \frac{\dot{k}_B}{k_B} = \frac{\dot{k}_S}{k_S} = 0$, and thus:

$$c_S^* = \sigma \frac{\left((\sigma - n_S) + g_S^N\right)}{\theta \beta_S^h} k_S^*$$

$$c_B^* = \sigma \frac{\left((\sigma - n_B) + g_B^N\right)(1-\tau_B^v)}{(1-\theta)\beta_B^h} k_B^*$$

Remembering that at the stationary state, we can express $k_i(c_i, v_i)$ since g_i^N is constant:

$$k_B^* = \beta_B^h \frac{\left(1 - \frac{(1-\theta)(\sigma-1)}{\sigma} C_B^* - v_B^*\right)}{g_B^N}$$

$$k_S^* = \beta_S^h \frac{\left(1 - \frac{\theta(\sigma-1)}{\sigma} C_S^* - v_S^*\right)}{g_S^N}$$

$$g_S^N = \frac{\gamma(1-k_S)n_B + (1-\eta(1-k_B))n_S}{\eta\kappa_B(1-\gamma) + (1-\gamma k_S)(1-\eta)}$$

$$g_B^N = \frac{(1-\gamma k_S)n_B + \eta k_B n_S}{\eta\kappa_B(1-\gamma) + (1-\gamma k_S)(1-\eta)}$$

Thus the expression of k_i^* and c_i^* are the following:

$$k_S^* = \frac{\beta_S^h}{g_S^N \left(1 + (\sigma-1)\left(\frac{\rho - n_S}{g_S^N} + 1\right)(1-\tau_S^h) + \left(\frac{\beta_S^v(1-\tau_S^h)}{\beta_S^h(1-\tau_S^v)}\right)^{\frac{1}{1-\mu}} \right)}$$

$$k_B^* = \frac{\beta_B^h}{g_B^N\left(1+(\sigma-1)\left(\frac{\rho-n_B}{g_B^N}+1\right)(1-\tau_B^h)+\left(\frac{\beta_B^v(1-\tau_B^h)}{\beta_B^h(1-\tau_B^v)}\right)^{\frac{1}{1-\delta}}\right)}$$

$$c_B^* = \sigma\frac{(1-\tau_B^h)\left((\rho-n_B)+g_B^N\right)}{g_B^N(1-\theta)\left(1+(\sigma-1)\left(\frac{\rho-n_B}{g_B^N}+1\right)(1-\tau_B^h)+\left(\frac{\beta_B^v(1-\tau_B^h)}{\beta_B^h(1-\tau_B^v)}\right)^{\frac{1}{1-\delta}}\right)}$$

$$c_S^* = \sigma\frac{\left((\rho-n_S)+g_S^N\right)}{g_S^N\theta\left(1+(\sigma-1)\left(\frac{\rho-n_S}{g_S^N}+1\right)(1-\tau_S^h)+\left(\frac{\beta_S^v(1-\tau_S^h)}{\beta_S^h(1-\tau_S^v)}\right)^{\frac{1}{1-\mu}}\right)}$$