Article can be accessed online at http://www.publishingindia.com

On the Role of Fuzzy Sets and Systems in Managerial Decision Making

C. R. Bector*, S. K. Bhatt**, S. S. Appadoo***

Abstract

The present paper discusses the history and the logic behind the conception and development of fuzzy sets and systems and the fuzzy logic therein, and how its suitability was recognised in modeling the decision making problems in management science, finance, marketing, and supply chain management, among others. Lately, fuzzy models are being used in multi-attribute decision making. Problems such as brand choice, consumer behaviour, head hunting in human resource management, in which human preferences are involved and have to be evaluated quantifiably to arrive at a tangible decision; fall in this category. The paper tries to explain the intricate fuzzy logic in a manager friendly language and how it is applied in the decision making process where linguistic ambiguity or imprecision is involved. We present the application of fuzzy logic and possibility theory in modeling and the solutions of business problems in various fields such as Finance, marketing, supply chain among others.

Keywords: Fuzzy Number, Fuzzy Mathematics and Fuzzy Logic, Possibility Theory, A-Cuts, Fuzzy Equations, Fuzzy Decisions, Possibilistic Mean and Variance

Introduction

We are living in the age of information revolution that has forced the world of business, industry, finance, and management to be sharp and efficient in sorting out the huge data, and in decision making. The recent incident on the Wall Street on May 6, 2010 is a reminder of the challenges where the Dow Jones average plunged by about a 1000 points within a three hours of trading. It caught Wall Street off guard and the experts attributed it to a computer glitch called the 'big finger" by which an "m" may have been typed as a "b"; even as an enquiry has been set out. The decision making arena not only deals with concrete and certain data but has to face the factors caused by human behaviour and the failings and the successes thereof. A case in point is the Bernie Madoff's Ponzi scheme in which he is said to have skimmed his investors of around 65 billion dollars. Much of it has given rise to the theory of neural finance which in a way challenges the efficient market hypothesis. Many attribute this to the "herd mentality" among investors.

We humans do not always make rational decisions. In the month of May 2010, Wall Street investors were indulging in what is known as "naked selling" that defies any logic, data or human behaviour. On many occasions we take decisions on "gut feeling", for instance, an entrepreneur was boasting when asked about the criteria he uses in buying a company and whether he uses the market position of the company or its cash flows etc. as taught to a Business student. The fact that the rational theory of Kahneman and the irrational theory (behavioural economics) of Gary Becker, find equal importance in the decision making discourse is highlighted by the Nobel Prizes in economics awarded to both.

In the present day deluge of information, it is inevitable for the researchers in decision making, especially in management science, that the relevance of information, its measure as well as its meaning are taken into account

* Professor Emeritus, Department of Business Administration, Asper School of Business, University of Manitoba, Winnipeg, Canada

** Professor Emeritus, Department of Supply Chain Management, Asper School of Business, University of Manitoba, Winnipeg, Canada. Email: skbhatt@cc.umanitoba.ca

*** Assistant Professor, Department of Supply Chain Management, Asper School of Business, University of Manitoba, Winnipeg, Canada. Email: ss.appadoo@umanitoba.ca

in the decision making process. The situations where the proper information is being avoided or not being used, has to be pointed out so that the consequent risks and imprecision can be handled. In what follows next, we wish to revisit the situations where we perform the decision making under - (1) certainty, (2) risk, (3) uncertainty, and (4) fuzziness (caused by linguistic ambiguity or vagueness or imprecision).

Every decision problem consists of a set of parameters that are often retrievable from the data taken from the organisation, and a set of (control) variables or activity levels. It is the nature of parameters that defines the kind of decision problem it is. If we are securing a housing loan from a bank and the bank states that the interest rate is 4.25% over 30 years on 200,000 dollars, payable monthly; we know that the three parameters- interest rate, amortization period of borrowing and the amount are known for certain or have crisp values, so to find out the monthly installments- the control variable, is a decision making problem with certainty.

Decision making under risk occurs when we estimate the present value of future cash flows or design an investment portfolio. The risk involved is estimated by the time value of money expressed as interest rate, inflation or deflation or depression, or variance derived from the data. Even though the variance of the sample data is not an unbiased estimate of the future variance, one has to make do with it in the absence of anything else. Here the objective is to minimise the risk which is measured as the squared deviation from the expected value. In many situations, the call is to minimise the downside risk when the inflows are below expected value.

In many business situations, where the parameters are dependent on uncertain events, we are dealing with decision making under uncertainty. Say a Ford dealer is placing an order for the upcoming year for the Ford Focus model. The order is to be placed in October 2010 for the Christmas season this year and for the New Year. He has to pay the total cost in advance. How many cars are to be ordered? Here the parameter represented by demand is uncertain and depends upon the economic upturn or down turn, competition and many such factors that are not in the dealer's control. His decision alternatives may be to order 500, 750 or 1000 cars and the state of nature may be whether the economy is unchanged (yielding a demand of 750 cars), may be weak (500 cars) or may improve (1000 cars). Here one has to work on the probability estimates of the state of nature resulting in a decision making under uncertainty.

Lastly, we describe the decision making under fuzzy environment. Suppose a decision maker (DM) is to buy a laptop. He has, among several, settled down on two brands: IBM and Toshiba. Of course, this choice has been arrived at through the deliberation on several criteria such as cost, weight, processor, battery life and after sales support (Bhatt et al, 2009). The DM says, "I prefer Toshiba over IBM". Here the word "prefer" introduces an ambiguity in terms of semantics regarding its meaning vis-à-vis criteria, and imprecision in terms of its syntax, grammar or degree as to whether it is most preferred, or more preferred and how much. This situation induces fuzziness in the decision making process and this linguistic variable is called the fuzzy variable.

Zadeh (1999), the author of the fuzzy concept, asserts that the pioneering work of Wiener and Shannon on the theory of communication focused on communication and the measure of information, and was statistical in nature. He theorizes that the main concern is the meaning of information – rather than its measure- and the proper framework of information is possibilistic, arising from fuzzy interpretation of the meaning of the linguistic variable. We, now introduce the basic definitions, algebra of fuzzy numbers and the related possibility theory. For basic definitions and algebra of fuzzy numbers, we refer to the seminal papers: Zadeh (1965), Bellman & Zadeh (1970), Zimmerman (1983), and Dubois & Prade (1989). For discussion on possibility theory, we refer to: Zadeh (1999), Carlsson & Fuller (2001), and Fuller & Majlender (2003). For further understanding of the literature of fuzzy numbers, we refer the readers to the excellent exposition of these concepts in the following books: Bector & Chandra (2005), Kaufmann & Gupta (1984), Klir & Youn (1995), Bojadziev & Bojadziev (1997), and Sukowa (1993) among others.

Preliminaries

The idea of a fuzzy number or a fuzzy set or a fuzzy variable germinates from the way a subset is defined in the theory of sets. Suppose R is the set of all MBA students in a university. Let X denotes the set of all female MBA students in the MB A class, which has 8 female and

22 male students. We can say that X is a subset of R. This can be described as X= {x ε R/ x is female}. The same can be described as a membership function (known as the characteristic function) f_X: R → {0, 1}, where $f_X(x)$ = 1 if x is a female student or 0 otherwise. Here the membership function is dichotomous in that it provides the membership value of 1 to every female MBA student and 0 to every male MBA student and nothing in-between. Now suppose we want to define the set A of "tall" female students. This set A definitely will be a subset of X but the qualification of being tall is not a crisp statement as against the set X was a crisp subset of the set R. It has introduced an ambiguity and imprecision. Suppose we collect the information on the heights of the female MBA students in descending order as below. One may provide a membership to define the set of tall female MBA students that would take values between 0 and 1.

Female student	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Height	5'8"	5'7"	5'6"	5'5"	5'4"	5'3"	5'2"	5'1"
Membership $\mu_A(x)$	1	1	0.8	0.7	0.5	0.3	0.1	0

The set A of tall female MBA students is called the fuzzy set with the membership function μ_A for each element of set X. The set A can be described as:

A= {(x,μ_A(x))/ xε X}or listed as A= {x/ μ_A(x)}= {5'8"/1, 5'7"/1, 5'6"/0.8, 5'5"/0.7, 5'4"/0.5, 5'3"/0.3, 5'2"/0.1, 5'1"/0}.

Thus a fuzzy set A has two parts, a crisp set X and a membership function μ_A that takes real values between 0 and 1 and specifies the grade or degree to which x in X belongs to the fuzzy set A. Note that the assignment of membership function can vary from individual DM to others. In case of a crisp set, the membership function takes only two values, 0 or 1. In the case of a fuzzy set, the membership function moves progressively from 1 to 0 or other way around. When the membership function $\mu_A$is known, we call the fuzzy set A, a fuzzy number (with some conditions). When the membership function is to be determined to conform to the given situation at hand, we call the set A, a fuzzy variable.

As stated earlier, the linguistic variable creates ambiguity and imprecision that can cause one to have many membership functions to represent the same fuzzy structure. Example: find a number close to 5. For X we may choose an interval on real line: xε [0,10] and $\mu_A(x)$= 1/ [1 + a(x – 5)^b], where one can choose a and b among natural numbers. As X is a continuous interval, μ_A will be a continuous function. Bellman & Zadeh (1970) give an example of a fuzzy goal expressed by the words "x should be substantially larger than 10" that can be expressed by the membership function on all real numbers as μ_A= 0 if x ≤10 and μ_A= 1/[1+ (x- 10)$^{-2}$]for x≥ 10.. The membership function for a fuzzy constraint "x should be approximately between 2 and 10" can be given by μ_A=1/ [1+ a(x- 6)^m] where one may choose appropriate natural numbers for a and m.

Measuring Fuzziness

The imprecision, vagueness or ambiguity created due to the meaning of the word can be expressed by fuzziness represented by a membership function. However, this membership function is not unique as we see in the above examples. Then, given two membership functions for the same fuzzy statement by two different decision makers (DMs), say about tallness of the female MBAs, is it possible to find out which of the two are fuzzier or crisper? Here again, Shanon's entropy measure of information can help. Suppose there are n outcomes with probabilities p_i, i= 1,2,..,n. Then the amount of uncertainty in this process is measured by -∑ p_i l_n p_i. Obviously, there is more uncertainty in rolling a die than tossing a coin. In fact this uncertainty becomes maximum when all p_i's are equal. Suppose, we have two fuzzy numbers A= {(x,μ_A(x))/ x ε X} and B= {(x,μ_B(x))/ x ε X}. Suppose, we define the entropy of A as:

H (A) = -∑ $\mu_A(x_i)$ l_n $\mu_A(x_i)$, then [de Luca and Termini (1972)], the amount of fuzziness in the fuzzy set A, denoted by d (A) = H(A) + H(A^-). Recall that A^- was the fuzzy complementation of the Fuzzy set A. Then by calculating d (A) and d (B), we would be able to see, which of the two fuzzy sets are crispier.

Now we formalize some definitions relating to the fuzzy sets and fuzzy numbers. For notational ease, we will use μ in place of μ_A.

Definition 1: Fuzzy set A in X $\subset \Re$, the set of real numbers, is a set of ordered pairs A ={(x, μ(x) : x ε X}, where μ(x) is the membership function or grade of membership, or degree of compatibility or degree of truth of x ε X which maps x ε X on the real interval [0, 1].

Definition 2: If Sup $\mu(x)=1$, for all $x \in \Re$, then the fuzzy set A is called a normal fuzzy set in $\Re$.

Definition 3: The crisp set of elements that belong to the fuzzy set A, at least to the degree α is called the α-level set (or α–cut), i.e. $A(\alpha) = \{x \in X \mid \mu(x) \geq \alpha,\ \alpha \in \Re^+\}$. If the set $A'(\alpha) = \{x \in X \mid \mu(x) > \alpha,\ \alpha \in \Re^+\}$, then $A'(\alpha)$ is called strong α-level set (or strong α–cut).

Definition 4: A fuzzy set A is said to be a convex set if $\mu(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu(x_1), \mu(x_2))$, $x_1, x_2 \in X$, $\lambda \in [0,1]$. Alternatively, a fuzzy set A is convex if its every α-level sets is a convex set.

Remark: If X is an interval $\subset \Re$ and $\mu(x)$ is a quasi-concave function, then every α–cut will be a convex set.

Definition 5: A fuzzy set A, which is both convex and normal, is defined to be a fuzzy number on the universal set $\Re$.

Definition 6: A fuzzy number is completely known by its α-cuts that introduces a new definition of a fuzzy number being represented by the interval of confidence $(a_1(\alpha), a_2(\alpha))$, where $a_1(\alpha)$ and $a_2(\alpha)$ are the values in A for which a given α cuts the graph of the membership function (Kaufmann & Gupta,1991).

Definition 7: A Triangular Fuzzy Number (T.F.N.) can be represented completely by a triplet $A = [a_1, a_2, a_3]$, where $a_1 \leq a_2 \leq a_3 \in \Re$ with membership function

$$\mu(x)\, as\, \mu(x) = \begin{cases} 0 & x \leq a_1 \\ \dfrac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2 \\ \dfrac{x-a_3}{a_2-a_3} & a_2 \leq x \leq a_3 \\ 0 & x \geq a_3 \end{cases} \quad (1.1)$$

We can define the interval of confidence at level αas:

$A(\alpha) = [a_1(\alpha), a_2(\alpha)]$ We may thus characterize the T.F.N. $[a_1, a_2, a_3]$ as,

$$A(\alpha) = [a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)],\ \forall \alpha \in (0,1]\,. \quad (1.2)$$

Definition 8: A Trapezoidal Fuzzy Number (Tr.F.N.) can be represented completely by a quadruplet A= $[a_1, a_2, a_3, a_4]$ where $a_1 \leq a_2 \leq a_3 \leq a_4 \in \Re$ with membership function $\mu(x)$ as

$$\mu(x) = \begin{cases} 0 & x \leq a_1 \\ \dfrac{x-a_1}{a_2-a_1} & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \dfrac{x-a_4}{a_3-a_4} & a_3 \leq x \leq a_4 \\ 0 & x \geq a_4 \end{cases} \quad (1.3)$$

Alternatively defining the interval of confidence at level α as

$A(\alpha) = [a_1(\alpha), a_2(\alpha)]$

we characterize the T_r.F.N.. $[a_1, a_2, a_3]$ as:

$$A(\alpha) = [a_1 + \alpha(a_2 - a_1), a_4 + \alpha(a_3 - a_4)],\ \forall \alpha \in (0,1]. \quad (1.4)$$

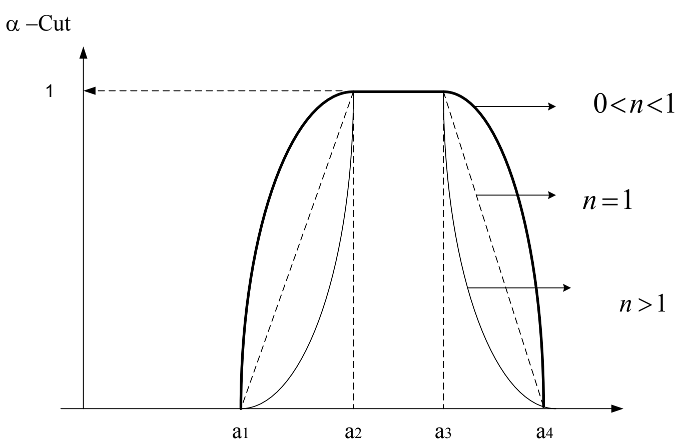
Definition 9: A fuzzy number $A = [a_1, a_2, a_3, a_4]_n$ can be expressed for all $x \in \Re$ in the form

$$A(x) = \begin{cases} g(x) & when \quad x \in [a_1, a_2] \\ 1 & when \quad x \in [a_2, a_3] \\ h(x) & when \quad x \in [a_3, a_4] \\ 0 & otherwise \end{cases} \quad (1.5)$$

where, a_1, a_2, a_3, a_4 are real numbers such that $a_1 < a_2 < a_3 < a_4$, g is a real valued function that is increasing and right continuous and h is a real valued function that is decreasing and left continuous. A fuzzy number A [(we call it an adaptive fuzzy number), Appadoo, 2006], with shape functions g and h defined by

$$g(x) = \left(\frac{x-a_1}{a_2-a_1}\right)^n,\ h(x) = \left(\frac{a_4-x}{a_4-a_3}\right)^n \quad (1.6)$$

respectively, where $n > 0$, is denoted by $A = [a_1, a_2, a_3, a_4]_n$. If $n = 1$, we simply write



$A=[a_1,a_2,a_3,a_4]$, which is known as a trapezoidal fuzzy number, and further if a_3=a_2 and it is a T.F.N. If $n \neq 1$, a fuzzy number $A^*=[a_1,a_2,a_3,a_4]_n$ is a modification of a trapezoidal fuzzy number $A=[a_1,a_2,a_3,a_4]$. If n > 1, then A^* is a concentration of A^*. If 0 < n < 1, then A^* is a dilation of A. Each fuzzy number A described above has the following α-level sets (α-cuts) $A(\alpha)=[a_1(\alpha),a_2(\alpha)]$, $a_1(\alpha),\ a_2(\alpha) \in \Re, \alpha \in [0,1]$

$$A(\alpha)=\left[g^{-1}(\alpha),h^{-1}(\alpha)\right],$$

$$A(\alpha=1)=[a_2,a_3],\ A(\alpha=0)=[a_1,a_4]$$

If $A=[a_1,a_2,a_3,a_4]_n$ then for all $\alpha \in [0,1]$,

$$A(\alpha)=\left[a_1+\alpha^{\frac{1}{n}}(a_2-a_1),\ a_4-\alpha^{\frac{1}{n}}(a_4-a_3)\right]$$

Fuzzy Algebra

As we stated above that the fuzzy sets or fuzzy numbers have been defined in two different ways even though they are analogous. One is purely by looking at the membership function of the fuzzy number and the other is by looking at the confidence interval. This gives rise to two schools for algebraic formulation, one led by Zadeh and the other led by Kaufmann & Gupta. With the membership function in mind, the algebra mirrors the algebra of sets with set inclusion and union and intersection of sets. That is the Zadeh school approach. Kaufmann & Gupta School looks at the fuzzy sets as interval of confidence and use addition and multiplication of intervals. We will focus on the former approach.

Consider the fuzzy sets A and B in the universal set R, so that

A = {{(x,μ_A(x))/ xε R}, μ_A(x) ε [0,1]} and B = {{(x,μ_B(x)/ xε R)}, μ_B(x) ε [0,1]}. Then, pertaining to these two fuzzy sets, we define;

Equality: The fuzzy sets A and B are equal, denoted by A = B, if and only if, for every x in R, $\mu_A(x) = \mu_B(x)$

Subset: A fuzzy set A is a subset of the fuzzy set B, denoted by A ⊆ B, if, for every x in R, $\mu_A(x) \leq \mu_B(x)$ and a proper subset if A is a subset of B and A≠ B and $\mu_A(x) < \mu_B(x)$ for at least one x in R

Complementation: The complementary set of a fuzzy set A denoted by A^- is defined by the membership function: $\mu_{A^-}(x) = 1 - \mu_A(x)$ or $\mu_{A^-}(x) + \mu_A(x) = 1$, x in R.

The membership function $\mu_{A^-}(x)$ is symmetric to $\mu_A(x)$ with respect to the line μ= 0.5.

Intersection: The operation of intersection of A and B, denoted by A∩B is induced bydefining the membership function of A∩B as $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$, xε R

Union: The union of two fuzzy sets A and B, denoted by AUB, is defined by the membership function $\mu_{AUB}(x)$ = max {$\mu_A(x)$, $\mu_B(x)$}, xε R.

A major difference between fuzzy sets and crisp sets is highlighted by the law of excluded middle. In the crisp set theory, A∩A^- = φ, but if A is a fuzzy set and the fuzzy complement is A^-,then in general, = min{$\mu_A(x)$, $\mu_B(x)$} ≠ 0 for some xε R.

The Possibility Theory

As stated earlier, Zadeh's arguments for providing a possiblistic structure in explaining the uncertainty due to imprecision as opposed to probability theory, lies in disseminating the meaning of the information rather than its measure. Just as the probability distribution and its laws use the algebra of sets, the possibility distribution and its laws are described by the membership function of a fuzzy set and the fuzzy set algebra.

Here is a well known example, given by Zadeh (1999), of the situation that provides a probabilistic and a possibilistic interpretation. Recall that the probability of an event identified as a (crisp) subset A of the universal set R, is defined by a function P(A)ε [0,1], with following rules:

i. 0 ≤ P(A)≤1
ii. P(R) =1
iii. If A and B are two sets, then P(AUB) = P(A) + P(B)- P(A∩B) and P(A∩B) = P(A) P(B).

In the realm of fuzzy sets and its algebra, it is natural that the possibility of an event, described by a fuzzy set A, would be determined by its membership function. If the fuzzy set is A= {(x, μ_A(x)/ x ε R}, then {μ_A(x)/ x ε R} can be looked upon as the possibility distribution of A andΠ(A) = max { μ_A(x)/ x ε R}.Following, (Zadeh, 1999), we may lay down the rules of the Possibility. LetA = {{(x,μ_A(x))/ xε R}, μ_A(x) ε [0,1]} and B = {{(x,μ_B(x)/ xε R)}, μ_B(x) ε [0,1]}be two fuzzy sets then,

i. $\Pi(\varphi) = 0$ and $0 \le \Pi(A) \le 1$, where φ is a fuzzy set in which the membership of every element is 0.

ii. $\Pi(R) = 1$ and

iii. $\Pi(A \cup B) = \text{Max}\{\mu_A(x), \mu_B(x)\}$ and $\Pi(A \cap B) = \text{Min}\{\mu_A(x), \mu_B(x)\}$

To understand the difference between probability theory and possibility theory and the distribution thereof, we cite an excellent example given by Zadeh (1999). Consider the statement "Sam ate x eggs in the breakfast", with x taking values in U = {1, 2, 3, 4….}, We may associate a possibility distribution with x by interpreting $\Pi_x(u)$as the degree of ease with which Sam can eat u eggs. We may also associate a probability distribution:$P_x(u)$ with x as the probability that Sam will eat x = u eggs in the breakfast. One may arrive at these assignments as shown in Table 1.

Table 1: Difference between Probability Theory and Possibility Theory

x	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	
$\Pi_x(u)$	1	1	1		1	0.8	0.6	0.4	0.2
$P_x(u)$	0.1	0.8	0.1	0	0	0	0	0	

Here, it is observed that, whereas the possibility that Sam may eat 3 eggs for breakfast is 1, probability of him doing so is quite small i.e. 0.1. Thus a high degree of possibility does not imply a high degree of probability, nor does a low degree of probability imply a low degree of possibility. Zadeh calls this heuristic connection between possibility and probability as the possibility/ probability consistency principle.

When we allow the fuzzy set A represented by its α- cuts in the form of an interval, Carlson and Fuller (2001), used it to define the mean and variance of the possibilistic distribution of a fuzzy number. Carlsson and Fuller (2001) first defined the crisp lower possibilistic mean value of a fuzzy number $E_L(A)$ and the crisp upper possibilistic mean value $E_R(A)$ of a fuzzy number A defined by $A = [a_1(\alpha), a_2(\alpha)]$, $\alpha \in [0,1]$ as follows.

$$E_L(A) = 2\int_0^1 \alpha\, a_1(\alpha)d\alpha, \quad (2.1)$$

$$E_R(A) = 2\int_0^1 \alpha\, a_2(\alpha)d\alpha, \quad (2.2)$$

The interval value possibilistic mean $IVPM(A)$ of a fuzzy number A is defined as

$$IVPM(A) = [E_L(A), E_R(A)] = \left[2\int_0^1 \alpha\, a_1(\alpha)d\alpha,\ 2\int_0^1 \alpha\, a_2(\alpha)d\alpha\right]. \quad (2.3)$$

As in Carlson and Fuller (2001), crisp possibilistic mean $E(A)$ and possibilistic variance $\text{var}(A)$ of a fuzzy number A, defined by $A = [a_1(\alpha), a_2(\alpha)]$, $\alpha \in [0,1]$ are given below.

$$E(A) = \frac{E_L(A) + E_L(A)}{2} = \int_0^1 \alpha(a_1(\alpha) + a_2(\alpha))d\alpha \quad (2.4)$$

where $E(A)$ is the level-weighted average of the arithmetic means of all α-level sets.

$$Var(A) = \int_0^1 Pos[A \le a_1]\left(\left[\frac{a_1(\alpha) + a_2(\alpha)}{2} - a_1(\alpha)\right]^2\right)d\alpha + \int_0^1 Pos[A \ge a_2]\left(\left[\frac{a_1(\alpha) + a_2(\alpha)}{2} - a_2(\alpha)\right]^2\right)d\alpha$$

$$= \int_0^1 \frac{1}{2}(a_1(\alpha) + a_2(\alpha))^2\ \alpha\ d\alpha \quad (2.5)$$

Here the variance of a fuzzy number A is the expected value of the squared deviations between the arithmetic mean and the endpoints of its level sets.

Let $A \text{ and } B \in \Im$ (the set of fuzzy numbers) be two fuzzy numbers with $A(\alpha) = [a_1(\alpha), a_2(\alpha)]$ and $B(\alpha) = [b_1(\alpha), b_2(\alpha)]$,

$\alpha \in [0,1]$. Goetschel and Voxman (1986) introduced a method for ranking fuzzy numbers as

$$A \le B \iff \int_0^1 (a_1(\alpha) + a_2(\alpha))\alpha\, d\alpha \le \int_0^1 (b_1(\alpha) + b_2(\alpha))\alpha\, d\alpha \quad (2.6)$$

As pointed out by Goetschel and Voxman (1986), the definition given in (2.6) for ordering fuzzy numbers was motivated by the desire to give less importance to the lower levels of fuzzy numbers. For a Tr.F.N. A, the possibilistic mean value is given by

$$E(A) = \frac{1}{3}(a_2 + a_3) + \frac{1}{6}(a_1 + a_4) \quad (2.7)$$

The possibilistic variance of a Tr.F.N is given by,

$$Var(A) = \frac{(a^2{}_3 + a^2{}_2)}{8} + \frac{(a^2{}_4 + a^2{}_1)}{24} + \frac{(a_3a_4 - a_3a_1 - a_4a_2 - a_4a_1 + a_2a_1)}{12} - \frac{a_3a_2}{4}$$

On the other hand if A is a T.F.N. then the possibilistic mean and possibilistic variance are given by

$$E(A)=\frac{2a_2}{3}+\frac{(a_1+a_3)}{6} \text{ and } Var(A)=\frac{\left(a^2{}_3+a^2{}_1\right)-2a_3a_1}{24}$$

Thavaneswaran, Thiagarajah & Appadoo (2007) have extended these results to obtain higher order moments about mean in order to be able to define the possibilistic analogues of skewness and kurtosis as in probability theory. (See also Appadoo, Bhatt & Bector, 2008).

Fuzzy Theory to Handle Uncertainty

Initially, the concept of fuzzy numbers was to address linguistic ambiguity and imprecision. Later, fuzzy modeling has been used to address uncertainty of events involved. For example, in PERT/CPM models, the time estimates for an activity is calculated as a β-distribution of three numbers, a- the pessimistic, m- the most likely and b-the optimistic estimate. It becomes a tedious probabilistic computation to develop a sound theory. In fuzzy modeling, it naturally lends to a triangular fuzzy estimate of time of the activity. There are many examples of application of fuzzy algebra in Project Planning(See Sutardi, Bector & Goulter,1995; Long &Ohsato,2008 among others)

In other situations, if we only know the mean and variance of the data, as in the case of statistical quality control that allows the production measure to remain within $\mu \pm 3\sigma$ limit, the process can be represented by a Gaussian fuzzy number whose membership function needs only the mean and the variance to define it. The standard Gaussian fuzzy number (Lowen & Verschoren, 2008), is represented by A = {x: μ, σ,} with membership function as:

$$\mu_A(x)= \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

where μ is the mean, and σ is the standard deviation estimated from the data.

Applications in Management

Supply Chain Management

Supply chain is a process in which goods/ information flow from the vendors to the customers through vendor→ manufacturer→ logistics/transportation→retailers→consumers. One of the important problems is the supplier selection problem. The objective of the vendor selection problem is to choose from a potential list of vendor candidates, the ones that best suit the company's interest. At the same time, the problem also should determine how many suppliers the company needs, and how much business has to be allocated to each supplier. Normally, the supplier selection problem involves attaining multiple objectives such as: supplier reputation, price, quality, delivery performance (DP), and after sale customer service (CS)at the same time; therefore, the management science models used to address this problem tend to have multi-criteria objectives. The most representative solution methods that fit this condition are multi-objective programming. Recently, Appadoo, Bhatt, Bector & Chandra (2007), and Appadoo *et al*. (2009) used the fuzzy TOPSIS method for the multi-criteria supplier selection problem for an individual and a group decision making situation. For each product, the suppliers were evaluated for the criteria such as: profile, technology, quality, delivery and flexibility.

Inventory Management

Inventories exist in every part of a supply chain. These may be with vendors, with manufactures on the shop floor, on the transportation, and with the retailers. A basic problem in inventory management is how much and when to order to minimize the overall costs of stocking (C_s) and ordering (C_o)(shortage or other costs can be dealt in more general settings). For a simple question with known demand D, the optimal (economic) order quantity commonly known as EOQ = $\sqrt{(2DC_s/ C_o)}$. If the parameters C_s and C_o are not known for certain, many researchers have developed the inventory solutions by taking C_s and C_o as fuzzy parameters. Some of these are by Roy & Maiti (1997), Lee &Yao (1998) and Chang (2004). More recently, Appadoo, Bhatt, Bector & Sharma (2008) have developed the most general form of EOQ under fuzzy environment.

Fuzzy Linear Programming

Linear programming is a mathematical modeling and solution of optimum allocation of resources in an economic unit in order to achieve a certain goal such as maximizing returns or minimizing costs. Though conceived by Dantzig & Kantorovich (Nobel Prize 1975) independently, the application of linear programming in economics, business and industries became inevitable after the publication of the book 'Linear Programming and Economic Analysis' by Samualson (Nobel prize 1970), Dorfman & Solow (Nobel prize 1987) in 1958, McGraw Hill. Tjalling Koopman (Nobel Prize 1975) used the duality structures of linear programming to assess the shadow prices of the resources and their implications in economic activity. There is hardly any area of management that has not been helped by linear programming to find solutions to the problems at hand, whether it is an investment problem in finance, developing an advertisement policy to maximize reach while remaining within budget, transportation of goods or a project planning problem. A typical linear programming problem (LP) is to maximize or minimize $Z= C^TX$, subject to linear constraints: $AX\leq B$ and the activity vector of decision variables $X\geq 0$, where C (cost), A (input) and B (capacity or requirement) parameters are in matrix forms. Bellman & Zadeh (1970) proposed a symmetric fuzzy linear programming in which both the objective value Z and the constraint capacity B are given a tolerance of an amount of p_i in percentage allowance. Later researchers considered the parameters C, A and B as fuzzy numbers and devised the solution procedure. To cite a few, refer to: Jemenez *et al*. (2007), Bector & Chandra (2002) and Bector *et al*. (2004).

Fuzzy Non Linear Programming

Two natural extensions of linear programming would be a): quadratic programming that optimizes a quadratic function with linear constraints or b): fractional programming where the objective function is a ratio of two linear functions. Two seminal papers by Harry Markowitz (Nobel Prize 1990) enhanced the further interest in quadratic programming, one, the paper on "Portfolio Selection"in 1952 and the other, an algorithm for quadratic programming in 1956. An excellent exposeon fuzzy quadratic programming can be found in Shiang –Tai Liu (2009). Bector, Bhatt & Sharma (2002), proposed a finite iteration technique for a fuzzy quadratic programming problem using the parametric method of Van de Panne (1966).

Modern day management is high on productivity, best practices and efficiency ratios. Linear fractional programming has been applied as data envelopment analysis (DEA) to productivity analysis, proposed by Charnes, Cooper & Rhodes (1978). A large volume of papers have appeared which either deal with fuzzy DEA or solution or duality of fuzzy fractional programming. We are quoting just four of them: Lertwosirikul *et al*. (2003), Yuh-Wen Chen, (2001), Mehra, Chandra & Bector, (2007) and Appadoo, Bector & Bhatt (2010).

Marketing

Consumer behaviour is a major part of the subject of marketing. The four parts of marketing strategy are: product development-price- promotion and distribution. Research in new product development identifies product design screening as critical to the product development process. Smimou, Bhatt & Dahl (2005) used a fuzzy version of Analytic Hierarchy process to select a car jack design out of five alternatives under three criteria of originality, appeal and effectiveness. This is a multi-attribute decision problem in a fuzzy environment. See also Verworn (2009) for more recent work.

Finance

The theory of net present value, the capital asset pricing model of Sharpe ((Nobel prize 1990), (also credited to Lintner, and Mossin who derived it independently), and the option pricing model of Black & Scholes, also attributed to Robert Merton, (the last two receiving the Nobel prizes in 1997), are three fundamental concepts in finance. Because of the risk and uncertainty of financial markets, early solutions of these problems involved statistics and probability. Now, with the development of fuzzy sets and systems, some applications of fuzzy models are forthcoming. Appadoo, Bhatt & Bector (2008) applied a fuzzy version of investments and returns to obtain net present value. Appadoo, Bector & Chandra (2005) applied a fuzzy increase and decrease in the price of stocks to develop a binomial model for option pricing on the lines of Muzzioli and Torricelli (2004). Bector, Bhatt & Appadoo (2003), extended the CAPM model under fuzzy information. Guo & Bhatt (1995) developed a fuzzy model for a Budgeting problem of project selection.

References

Appadoo, S. S. (2006). Pricing financial derivatives with fuzzy algebraic models: A theoretical and computational approach, *Ph.D. Thesis*, University of Manitoba, Winnipeg, Manitoba, Canada.

Appadoo, S. S., Bector, C. R., & Bhatt, S. K. (2010). A fuzzy linear fractional programming model. *Proceedings of the Administrative Sciences Association of Canada Conference,* Regina, 22-25.

Appadoo, S. S., Bector, C. R., & Chandra, S. (2005). Binomial option pricing model using O(2,2)- Tr. T.F.N. *Administrative Sciences Association of Canada Conference Proceedings*, Toronto, Ointario.

Appadoo, S. S., Bhatt, S. K., & Bector, C. R. (2008). Application of possibility theory to investment decisions. *Fuzzy Optimization and Decision Making*, 7, 35-57.

Appadoo, S. S., Bhatt, S. K., Bector, C. R., & Chandra, S. (2007). Possibility supported TOPSIS model with application to vendor selection problem in supply chain management. *Proceeding of the Administrative Sciences Association of Canada Conference, Ottawa*, Canada. Recipients of the "honorable Mention Award" at the conference

Appadoo, S. S., Bhatt, S. K., & Bector, C. R., & Sharma, V. N. (2008). A possibility assisted fuzzy EOQ inventory model. *Proceeding of the Administrative Sciences Association of Canada Conference*, Dalhousie, Canada.

Appadoo, S. S., Bhatt, S. K., Bector, C. R., & Sharma, V. N. (2009). "(φ, λ) Mixed Strategy for Group Multi-Attribute TOPSIS model with Application to Supplier Selection Problem. *Proceeding of the Administrative Sciences Association of Canada Conference* Niagara Falls, Canada.

Bector, C. R., Bhatt, S. K., & Appadoo, S. S. ((2003). Capital asset pricing model under fuzzy environment. *ASAC Conference Proceedings,* Halifax, Nova Scotia.

Bector, C. R., Bhatt, S. K., & Sharma, V.N. (2002). A finite Iteration Technique for a Fuzzy Quadratic Programming Problem" *7th International Symposium on Generalized Convexity/ Monotonicity*, Hanoi, Aug. 27-31, (Also chapter 4, Bector and Chandra (2005))

Bector, C. R., & Chandra, S. (2005). *Fuzzy mathematical Programming and Fuzzy matrix Games*, Springer, Berlin.

Bector, C. R., & Chandra, S. (2002). On duality on linear programming under fuzzy environment. *Fuzzy Sets and Systems,* 125, 317-325.

Bector, C. R., Chandra, S., & Vidyottama, V. (2004). Matrix games with fuzzy goals and fuzzy linear programming duality. *Fuzzy Optimization and Decision making,* 3, 255-269.

Bellman, R. E., & Zadeh, L. A. Decision-making in a fuzzy environment. *Management Science,* 17(4), B141-B164.

Bhatt, S. K., Bhatnagar, N., & Appadoo, S. S. (2009). *Presented at the Annual Meeting of the Society for Judgment and Decision Making,* 21-23, Boston, MA.

Bojadziev, G., & Bojadziev, M. (1997). Fuzzy logic for business, finance and management. *World Scientific,* Singapore.

Carlsson, C., & Fuller, R. (2001). On possibilstic mean value and variance of fuzzy numbers. *Fuzzy Sets and Systems*, 122, 315-326.

Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operations Research*, 12(6), 429-444.

Dubois, D., & Prade, H. (1989). Fuzzy sets, probability and measurement. *E. J. O. R.*, 40, 135-154.

Fuller, R., & Majlender, P. (2003).On weighted possibilistic mean and variance of fuzzy numbers. *Fuzzy sets and systems*, 136, 363-374.

Goetschel, R., & Voxman, W. (1986). Elementary fuzzy calculus. *Fuzzy Sets and Systems,* 18, 31-43.

Guo, T., & Bhatt, S. K. (1995). Fuzzy discrete multi-attribute utility approach to project selection, *Proceedings of Administrative Sciences Association of Canada*, Windsor, Ontario, Canada.

Kaufmann, A., & Gupta, M. M. (1991). *Introduction to fuzzy arithmetic, theory and applications*, Van Nostrand Reinhold.

Klir, G. H., & Bo, Y. (1995). *Fuzzy Sets and Fuzzy Logic-Theory and Applications,* Prentice Hall, N.J.

Lertworasirikul, S., Fang, S., Joins, J. A., & Nuttle, H. L.W. (2003).Fuzzy data envelopment analysis: A possibility approach. *Fuzzy Sets and Systems*, 139, 379-394.

Long, L. D., & Ohsato, A.(2008). Fuzzy critical chain method for project scheduling underresource constraints and uncertainty. *International Journal of Project management,* 26, 688-698.

Lowen, R., & Verschoren, A. (2008). Foundation of genetic optimization, 2, Application of fuzzy control, Genetic algorithm and neural networks, Springer.

De Luca, A., & Termini, S. (1972). A definition of a non probabilistic entropy in the setting of fuzzy set theory. *Information Control*, 20, 301-312.

Markovitz, H. M. (1952). Portfolio selection. *The Journal of Finance*, 12, 77-91.

Markovitz, H. M. (1956). The optimization of quadratic function subject to linear constraints. *Naval Research Logistic Quarterly*, 3, 111-133.

Mehra, A., Chandra, S., & Bector, C. R. (2007). Acceptable optimality in linear fractional programming with fuzzy coefficients. *Fuzzy Optimization & Decision Making,* 6, 5-16.

Muzzioli, S., & Torriceli, C. (2004). A multi period binomial model for pricing options in a vague world. *Journal of Economic Dynamics and Control,* 28, 861-887.

Liu, S. T. (2009). A revisit to quadratic programming with fuzzy parameters. *Chaos, Soliton, and Fractals,* 41, 1401-1407.

Smimou, K., Bhatt, S. K., & Dahl, D. W. (2005). New product ideas screening decision with approximate evaluations: Optimization approach. *Fuzzy Economic Review*, 10(2), 45-60.

Sukawa, M., (1993). *Fuzzy sets and Interactive Optimization*, Plenum Press, N.Y.

Sutardi, C., Bector, C. R., & Ian, G. (1995). Multi-objective water resource planning under budgetary uncertainty and fuzzy environment. *European Journal of Operational Research*, 82(3), 556-591.

Thavaneswaran, A., Thiagarajah, K., & Appadoo, S. S. (2007). Fuzzy Coefficient Valatility (FCV) models with applications. *Mathematical and Computer Modelling*, 45, 777-786.

Verworn, B. (2009). SA structural equation model of the impact of the "fuzzy front end" on the success of new product development. *Research Policy,* 38, 1571-1581

Yuh-Wen, C. (2001). Data envelopment analysis with fuzzy inputs and outputs. *IJQM*, 7(3), 165-176.

Zadeh, L.A. (1965). Fuzzy Sets. *Information Control*, 8.

Zadeh, L. A. (1999). Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems,* 100 Supplement, 9-34, Reprinted from the same journal, 1, (1978), 3-28.

Zimmerman, H. J., (1983). Fuzzy mathematical Programming. *Computers and Operations Research*, 10(4), 291-298.